Numbers and Sets (2018–19) Example Sheet 3 of 4

1. Use the inclusion-exclusion principle to count the number of primes less than 121.

2. How many subsets of $\{1, 2, \ldots, n\}$ are there of even size?

3. By suitably interpreting each side, or otherwise, establish the identities

$$\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n-1}{k} + \binom{n}{k} = \binom{n+1}{k+1}$$

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$$

4. By considering the number of ways to partition a set of order $2n$ into n parts of order 2, show that $(n+1)(n+2)\ldots(2n)$ is divisible by 2^n but not by 2^{n+1}.

5. In how many ways can $\{1, \ldots, n\}$ be written as the union of two sets? (Here, for example, $\{1, 2, 3, 4\} \cup \{4, 5\}$ and $\{4, 5\} \cup \{1, 2, 3, 4\}$ count as the same way of writing $\{1, 2, 3, 4, 5\}$.)

6. A *triomino* is an L-shaped pattern made from three square tiles. A $2^k \times 2^k$ chessboard, whose squares are the same size as the tiles, has one of its squares painted puce. Show that the chessboard can be covered with triominoes so that only the puce square is exposed.

7. Let A be a set of n positive integers. Show that every sequence of 2^n numbers taken from A contains a consecutive block of numbers whose product is a square. (For instance, 2,5,3,2,5,2,3,5 contains the block 5,3,2,5,2,3.)

8. Evaluate $a(4, 4)$ for the function $a(m, n)$, which is defined for integers $m, n \geq 0$ by

$$\begin{aligned} a(0, n) &= n + 1, \text{ if } n \geq 0; \\ a(m, 0) &= a(m-1, 1), \text{ if } m > 0; \\ a(m, n) &= a(m-1, a(m, n-1)), \text{ if } m > 0, \text{ and } n > 0\,. \end{aligned}$$

9. Using the least upper bound axiom, prove that there is a real number x satisfying $x^3 = 2$.

10. Prove that $\sqrt{2} + \sqrt{3}$ is irrational and algebraic. Do the same for $2^{1/3} + 2^{2/3}$.

11. Suppose that $x \in \mathbb{R}$ and $x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_0 = 0$, where $a_{n-1}, \ldots, a_0 \in \mathbb{Z}$. Prove that either x is an integer or it is irrational.

12. A real number $r = 0 \cdot d_1 d_2 d_3 \ldots$ is called *repetitive* if its decimal expansion contains arbitrarily long blocks that are the same; that is, for every k there exist distinct m and n such that $d_m = d_n$, $d_{m+1} = d_{n+1}$, $\ldots$, $d_{m+k} = d_{n+k}$. Prove that the square of a repetitive number is repetitive.

13. Show that $\sqrt[100]{\sqrt{3} + \sqrt{2}} + \sqrt[100]{\sqrt{3} - \sqrt{2}}$ is irrational.

***14.** Find a bijection $f : \mathbb{Q} \to \mathbb{Q} \setminus \{0\}$. Can f be strictly increasing (that is, $f(x) < f(y)$ whenever $x < y$)?