1. Does $3381x + 2646y = 21$ have an integer solution? Find the convergents to $\frac{152}{90}$. Find an integer solution to $152u + 90v = 2$. Find all such solutions.

2. Let $a, b, c, d \in \mathbb{N}$. Must the numbers $(a, b)(c, d)$ and (ac, bd) be equal? If not, must one be a factor of the other? If $(a, b) = (a, c) = 1$, must we have $(a, bc) = 1$?

3. Find integers x, y and z such that $56x + 63y + 72z = 1$.

4. Show that a number is divisible by 9 if, and only if, the sum of its digits is divisible by 9.

5. The *Fibonacci numbers* $F_0, F_1, F_2 \ldots$ are defined by $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$. Is F_{2018} even or odd? Is it a multiple of 3?

Show (by induction on k or otherwise) that $F_{n+k} = F_k F_{n+1} + F_{k-1} F_n$ for $k \geq 1$. Deduce that $(F_m, F_n) = (F_{m-n}, F_n)$, and thence that $(F_m, F_n) = F_{(m,n)}$.

6. Solve (i.e., find all solutions to) these congruences:-

(i) $77x \equiv 11 \pmod{40}$, (ii) $12y \equiv 30 \pmod{54}$,

(iii) $z \equiv 13 \pmod{21}$ and $3z \equiv 2 \pmod{17}$ simultaneously.

7. Without using a calculator, evaluate $20!21^{20} \pmod{23}$ and $17^{10000} \pmod{30}$.

8. By considering the n fractions $\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n}$, or otherwise, prove that $n = \sum_{d|n} \varphi(d)$.

9. An RSA encryption scheme (n, e) has modulus $n = 187$ and encoding exponent $e = 7$. Find a suitable decoding exponent d. Check your answer by encoding the number 35 and then decoding the result. *(Remember, no calculators!)*

10. Do there exist 100 consecutive natural numbers, each of which has a proper square factor?

11. Let p be a prime of the form $3k + 2$. Show that if $x^3 \equiv 1 \pmod{p}$ then $x \equiv 1 \pmod{p}$. Deduce that every number is a cube (mod p): i.e., $y^3 \equiv a \pmod{p}$ is soluble for all $a \in \mathbb{Z}$. Is the same ever true if p is of the form $3k + 1$?

12. The *repeat* of a natural number is obtained by writing it twice in a row (for example, the repeat of 356 is 356356). Is there a number whose repeat is a perfect square?

13. Is there a positive integer n for which $n^7 - 77$ is a Fibonacci number?

***14.** Let $a < b$ be distinct natural numbers. Prove that every block of b consecutive natural numbers contains two distinct numbers whose product is divisible by ab. Suppose now $a < b < c$. Must every block of c consecutive numbers contain three distinct numbers whose product is divisible by abc?