# Numbers and Sets (2018–19) Example Sheet 1 of 4

**1.** Between $0$ and $10$ there are four primes. Another example of two consecutive multiples of ten, between which there are four primes, is $10$ and $20$. Are there further examples?

**2.** If $n^2$ is a multiple of 3, must $n$ be a multiple of 3?

**3.** Write down the negations of the following assertions (where $m, n, a, b \in \mathbb{N}$):

(i) if I don't take the money then someone else will,

(ii) $\forall m\, \exists n\, \forall a\, \forall b\ \ (n \geq m)\ \wedge\ [(a = 1) \vee (b = 1) \vee (ab \neq n)]$ .

**4.** Find the largest product of (some not necessarily distinct) natural numbers whose sum is 100.

**5.** Prove that $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.

**6.** The *symmetric difference* $A \triangle B$ of two sets $A$ and $B$ is the set of elements that belong to exactly one of $A$ and $B$. Prove that $(A \triangle B) \triangle C = A \triangle (B \triangle C)$.

**7.** Let $A_1, A_2, A_3, \ldots$ be sets such that $A_1 \cap A_2 \cap \ldots \cap A_n \neq \emptyset$ holds for all $n$. Must it be that $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$?

**8.** Prove that $f \circ g$ is injective if $f$ and $g$ are injective. Does $f \circ g$ injective imply $f$ is injective? Does it imply $g$ is injective? What if we replace 'injective' by 'surjective' throughout?

**9.** Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4, 5\}$? How many functions $A \to B$ are there? How many are injections? Count the number of surjections $B \to A$.

**10.** Let $f : X \to Y$. Let $A, B \subset X$ and $C, D \subset Y$. For each of claims (a)–(f) below, give a proof or counter-example.

(a) $f(A \cup B) = f(A) \cup f(B)$ (b) $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$

(c) $f(A \cap B) = f(A) \cap f(B)$ (d) $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$

(e) $f^{-1}(f(A)) = A$ (f) $f(f^{-1}(C)) = C$

Show that each false claim can be made true by replacing '$=$' by either '$\subset$' or '$\supset$'.

**11.** Define a relation $R$ on $\mathbb{N}$ by $aRb$ if $a$ divides $b$ or $b$ divides $a$. Is $R$ an equivalence relation?

**12.** The relation $S$ *contains* the relation $R$ if $aSb$ whenever $aRb$. Let $R$ be the relation on $\mathbb{Z}$ '$aRb$ if $b = a + 3$'. How many equivalence relations on $\mathbb{Z}$ contain $R$?

**13.** Construct a function $f : \mathbb{R} \to \mathbb{R}$ that takes every value on every interval — in other words, for every $a < b$ and every $c$ there is an $x$ with $a < x < b$ such that $f(x) = c$.

$^*$**14.** Each of an infinite set of Trappist set theorists is going to a party, where each will receive a coloured hat, either red or blue. Each person will be able to see every hat but his own. After all hats are assigned, the set theorists must, simultaneously, each write down (in silence, obviously) a guess as to their own hat colour. You are asked to supply them with a strategy such that, should they follow it, only finitely many of them will guess wrongly. Can you?