Numbers and Sets (2017–18) Example Sheet 4 of 4

1. Prove that $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.

2. The *symmetric difference* $A \triangle B$ of two sets A and B is the set of elements that belong to exactly one of A and B. Express this in terms of $\cup$, $\cap$ and $\setminus$. Prove that $\triangle$ is associative.

3. Let $A_1, A_2, A_3, \ldots$ be sets such that $A_1 \cap A_2 \cap \ldots \cap A_n \neq \emptyset$ holds for all n. Must it be that $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$?

4. Prove that $f \circ g$ is injective if f and g are injective. Does $f \circ g$ injective imply f injective? Does it imply g injective? What if we replace 'injective' by 'surjective' everywhere?

5. Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4, 5\}$? How many functions $A \to B$ are there? How many are injections? Count the number of surjections $B \to A$.

6. Let $f : X \to Y$. Let $A, B \subset X$ and $C, D \subset Y$. For each of claims (a)–(f) below, give a proof or counter-example.

(a) $f(A \cup B) = f(A) \cup f(B)$ (b) $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$
(c) $f(A \cap B) = f(A) \cap f(B)$ (d) $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$
(e) $f^{-1}(f(A)) = A$ (f) $f(f^{-1}(C)) = C$

Show that each false claim can be made true by replacing '$=$' by either '$\subset$' or '$\supset$'.

7. Define a relation R on $\mathbb{N}$ by setting aRb if $a \mid b$ or $b \mid a$. Is R an equivalence relation?

8. The relation S *contains* the relation R if aSb whenever aRb. Let R be the relation on $\mathbb{Z}$ 'aRb if $b = a + 3$'. How many equivalence relations on $\mathbb{Z}$ contain R?

9. Show that any collection of pairwise disjoint discs in the plane is countable. What happens if we replace 'discs' by 'circles'?

10. Let $\mathcal{F}$ be the set of all finite subsets of $\mathbb{N}$. What goes wrong with the diagonal argument to show that $\mathcal{F}$ is uncountable? Show that, in fact, $\mathcal{F}$ is countable.

11. A function $f : \mathbb{N} \to \mathbb{N}$ is *increasing* if $f(n+1) \geq f(n)$ for all n and *decreasing* if $f(n+1) \leq f(n)$ for all n. Is the set of increasing functions countable or uncountable? What about the set of decreasing functions?

12. Find an injection $\mathbb{R}^2 \to \mathbb{R}$. Is there an injection from the set of all real sequences to $\mathbb{R}$?

13. Find a bijection $f : \mathbb{Q} \to \mathbb{Q} \setminus \{0\}$. Can f be strictly increasing (that is, $f(x) < f(y)$ whenever $x < y$)?

14. Let $S \subset \mathcal{P}\mathbb{N}$ be such that if $A, B \in S$ then $A \subset B$ or $B \subset A$. Can S be uncountable?

Is there an uncountable family $T \subset \mathcal{P}\mathbb{N}$ such that $A \cap B$ is finite for all distinct $A, B \in T$?

15. Each of an infinite sequence of Trappist set theorists is going to a party, where each will receive a coloured hat, either red or blue. Each person will be able to see every hat but his own. After all hats are assigned, the set theorists must, simultaneously, each write down (in silence, obviously) a guess as to their own hat colour. You are asked to supply them with a strategy such that, should they follow it, only finitely many of them will guess wrongly. Can you do it?