Numbers and Sets (2017–18) Example Sheet 3 of 4

1. Using the least upper bound axiom, prove that there is a real number x satisfying $x^3 = 2$.

2. Prove that $\sqrt{2} + \sqrt{3}$ is irrational and algebraic. Do the same for $2^{1/3} + 2^{2/3}$.

3. Suppose that $x \in \mathbb{R}$ and $x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_0 = 0$, where $a_{n-1}, \ldots, a_0 \in \mathbb{Z}$. Prove that either x is an integer or it is irrational.

4. Define a sequence $(x_n)_{n=1}^{\infty}$ by setting $x_1 = 1$ and $x_{n+1} = \frac{x_n}{1+\sqrt{x_n}}$ for all $n \geq 1$. Show that $(x_n)_{n=1}^{\infty}$ converges, and determine its limit.

5. Let $(a_n)_{n=1}^{\infty}$ be a sequence of reals. Show that if $(a_n)_{n=1}^{\infty}$ is convergent then we must have $a_n - a_{n-1} \to 0$. If $a_n - a_{n-1} \to 0$, must $(a_n)_{n=1}^{\infty}$ be convergent?

6. Which of the following sequences $(x_n)_{n=1}^{\infty}$ converge?

$$x_n = \frac{3n}{n+3} \qquad x_n = \frac{n^{100}}{2^n} \qquad x_n = \sqrt{n+1} - \sqrt{n} \qquad x_n = (n!)^{1/n}$$

7. Which of the following series converge?

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2} \qquad \sum_{n=1}^{\infty} \frac{n!}{n^n} \qquad \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+n}} \qquad \sum_{n=1}^{\infty}{}^{*} \frac{1}{n}$$

In the last case, the $*$ means omit all values of n which, when written in base 10, have some digit equal to 7.

8. Let $a_n \in \mathbb{R}$ and let $b_n = \frac{1}{n}\sum_{i=1}^{n} a_i$. Show that, if $a_n \to a$ as $n \to \infty$, then $b_n \to a$ also.

9. Let $\sum_{n=1}^{\infty} x_n$ be a divergent series, where $x_n > 0$ for all n. Show that there is a divergent series $\sum_{n=1}^{\infty} y_n$ with $y_n > 0$ for all n, such that $y_n/x_n \to 0$.

10. A real number $r = 0 \cdot d_1 d_2 d_3 \ldots$ is called *repetitive* if its decimal expansion contains arbitrarily long blocks that are the same; that is, for every k there exist distinct m and n such that $d_m = d_n,\ d_{m+1} = d_{n+1},\ \ldots,\ d_{m+k} = d_{n+k}$. Prove that the square of a repetitive number is repetitive.

11. Show that $\sqrt[100]{\sqrt{3}+\sqrt{2}} + \sqrt[100]{\sqrt{3}-\sqrt{2}}$ is irrational.

12. Let $\sum_{n=1}^{\infty} x_n$ be convergent. If $x_n > 0$ for all n, show that $\sum_{n=1}^{\infty} x_n^2$ also converges. What if sometimes $x_n < 0$? What are the corresponding answers for $\sum_{n=1}^{\infty} x_n^3$?

13. Construct a function $f : \mathbb{R} \to \mathbb{R}$ that takes every value on every interval — in other words, for every $a < b$ and every c there is an x with $a < x < b$ such that $f(x) = c$.