1. Show that a number is divisible by 9 if, and only if, the sum of its digits is divisible by 9.

2. The *Fibonacci numbers* $F_0, F_1, F_2 \ldots$ are defined by $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$. Is F_{2015} even or odd? Is it a multiple of 3?

Show (by induction on k or otherwise) that $F_{n+k} = F_k F_{n+1} + F_{k-1} F_n$ for $k \geq 1$. Deduce that $(F_m, F_n) = (F_{m-n}, F_n)$, and thence that $(F_m, F_n) = F_{(m,n)}$.

3. Solve (i.e., find all solutions of) these congruences:-

(i) $77x \equiv 11 \pmod{40}$, (ii) $12y \equiv 30 \pmod{54}$,

(iii) $z \equiv 13 \pmod{21}$ and $3z \equiv 2 \pmod{17}$ simultaneously.

4. Do there exist 100 consecutive natural numbers, each of which has a proper square factor?

5. Without using a calculator, evaluate $20!21^{20} \pmod{23}$ and $17^{10000} \pmod{30}$.

6. By considering the n fractions $\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n}$, or otherwise, prove that $n = \sum_{d|n} \varphi(d)$.

7. An RSA encryption scheme (n, e) has modulus $n = 187$ and encoding exponent $e = 7$. Find a suitable decoding exponent d. Check your answer by encoding the number 35 and then decoding the result. *(Remember, no calculators!)*

8. Let p be a prime of the form $3k + 2$. Show that if $x^3 \equiv 1 \pmod{p}$ then $x \equiv 1 \pmod{p}$. Deduce that every number is a cube (mod p): i.e., $y^3 \equiv a \pmod{p}$ is soluble for all $a \in \mathbb{Z}$. Is the same ever true if p is of the form $3k + 1$?

9. A *triomino* is an L-shaped pattern made from three square tiles. A $2^k \times 2^k$ chessboard, whose squares are the same size as the tiles, has one of its squares painted puce. Show that the chessboard can be covered with triominoes so that only the puce square is exposed.

10. Let A be a set of n positive integers. Show that every sequence of 2^n numbers taken from A contains a consecutive block of numbers whose product is a square. (For instance, 2,5,3,2,5,2,3,5 contains the block 5,3,2,5,2,3.)

11. Use the inclusion-exclusion principle to count the number of primes less than 121.

12. How many subsets of $\{1, 2, \ldots, n\}$ are there of even size?

13. By suitably interpreting each side, or otherwise, establish the identities

$$\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n-1}{k} + \binom{n}{k} = \binom{n+1}{k+1}$$

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$$

14. Show that $a^3 + b^5 = 7^{7^{7^7}}$ has no solution with $a, b \in \mathbb{Z}$.