Mich. 2016 **NUMBERS AND SETS—EXAMPLES 3** PAR

1. Prove carefully, using the least upper bound axiom, that there is a real number x satisfying $x^3 = 2$. Prove also that such an x must be irrational.

2. Prove that $\sqrt{2} + \sqrt{3}$ is irrational and algebraic.

3. Prove that $\log_2 3$ is irrational.

4. Let $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ be sequences of reals. Show that if $x_n \to 0$ and $y_n \to 0$ then $x_n y_n \to 0$. By considering $x_n - c$ and $y_n - d$, prove carefully that if $x_n \to c$ and $y_n \to d$ then $x_n y_n \to cd$.

5. Let $(x_n)_{n=1}^{\infty}$ be a convergent sequence of real numbers, and suppose there is a real number M such that for all n we have $x_n \leqslant M$. Prove that $\lim x_n \leqslant M$. If $x_n < M$ for all n, must we have $\lim x_n < M$?

6. Let $(x_n)_{n=1}^{\infty}$ be a sequence of reals. Show that if $(x_n)_{n=1}^{\infty}$ is convergent then $x_n - x_{n-1} \to 0$. If $x_n - x_{n-1} \to 0$, must $(x_n)_{n=1}^{\infty}$ be convergent?

7. Which of the following sequences $(x_n)_{n=1}^{\infty}$ converge?
(a) $x_n = \frac{3n}{n+3}$; (b) $x_n = \frac{n^{100}}{2^n}$; (c) $x_n = \sqrt{n+1} - \sqrt{n}$; (d) $x_n = (n!)^{1/n}$.

8. Which of the following series converge?
(a) $\sum_{n=1}^{\infty} \frac{1}{1+n^2}$; (b) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$; (c) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+n}}$.

9. Let $\sum_{n=1}^{\infty} x_n$ be a divergent series with $x_n > 0$ for all n. Show that there is a divergent series $\sum_{n=1}^{\infty} y_n$ with $y_n > 0$ for all n and $y_n / x_n \to 0$.

10. Show that it $\sum_{n=1}^{\infty} x_n$ is a convergent series of positive real numbers then $\sum_{n=1}^{\infty} x_n^2$ is also convergent. What happens if we do not insist that the x_n be positive?

11. If $\sum_{n=1}^{\infty} x_n$ is a convergent series of reals, must $\sum_{n=1}^{\infty} x_n^3$ be convergent?

12. Show that $\sqrt[100]{\sqrt{3} + \sqrt{2}} + \sqrt[100]{\sqrt{3} - \sqrt{2}}$ is irrational.