Mich. 2016 **NUMBERS AND SETS—EXAMPLES 2** PAR

1a) Find the highest common factor of 12345 and 54321.
b) Find integers x and y with $152x + 90y = 2$. Then find all pairs of integers x and y with $152x + 90y = 2$.
c) Do there exist integers x and y with $3381x + 2646y = 21$?

2. Let n be a natural number written in decimal notation as '$d_k d_{k-1} \ldots d_0$'. Show that n is a multiple of 11 if and only if $d_0 - d_1 + d_2 - \ldots + (-1)^k d_k$ is a multiple of 11.

3. Prove that if $a|bc$ and a is coprime to b then $a|c$; give two proofs, one based on Euclid's algorithm and one based on uniqueness of prime factorization.

4. Find all solutions of the congruences:
(i) $7w \equiv 77\,(40)$;
(ii) $12x \equiv 30\,(54)$;
(iii) $3y \equiv 2\,(17)$ and $4y \equiv 3\,(19)$ (simultaneously);
(iv) $z \equiv 2\,(3)$, $z \equiv 3\,(4)$, $z \equiv 4\,(7)$ and $z \equiv 5\,(10)$ (simultaneously).

5. Without using a calculator, find the remainder when $20!21^{20}$ is divided by 23, and the remainder when 17^{10000} is divided by 30.

6. Explain (without electronic assistance) why 23 cannot divide $10^{881} - 1$.

7. Let p be a prime of the form $3k + 2$. Show that if $x^3 \equiv 1\,(p)$ then $x \equiv 1\,(p)$. Deduce, or prove directly, that every integer is a cube modulo p; that is, prove that for every integer y there is an integer a with $a^3 \equiv y\,(p)$. Is the same ever true if p is of the form $3k + 1$?

8. By considering numbers of the form $(2p_1p_2 \ldots p_k)^2 + 1$, prove that there are infinitely many primes of the form $4n + 1$.

9. Do there exist 100 consecutive natural numbers each of which is divisible by a square number other than 1?

10. Do there exist integers a and b with $a^3 + b^5 = 7^{7^{7^7}}$?

11. The *repeat* of a natural number is obtained by writing it twice in a row (so, for example, the repeat of 254 is 254254). Is there a natural number whose repeat is a square number?

12. Let a and b be distinct natural numbers with $a < b$. Prove that every block of b consecutive natural numbers contains two distinct numbers whose product is a multiple of ab. If a, b and c are distinct natural numbers with $a < b < c$, must every block of c consecutive natural numbers contain three distinct numbers whose product is a multiple of abc?