Mich. 2016 **NUMBERS AND SETS – EXAMPLES 1** PAR

0. If you want to think more about how we develop properties of number systems from the axioms then look at the questions on the back of the sheet (after working seriously at what's on the front). If not, then get on with the questions below and don't turn over.

1. The numbers 3, 5 and 7 are all prime; does it ever happen again that three numbers of the form n, $n+2$, $n+4$ are all prime?

2. There are four primes between 0 and 10 and between 10 and 20. Does it ever happen again that there are four primes between two consecutive multiples of 10?

3. Consider the sequence 41, 43, 47, 53, 61, 71, 83, ... Are all of these numbers prime?

4. Show that $2^{19} + 5^{40}$ is not prime. Show also that $2^{91} - 1$ is not prime.

5. If n^2 is a multiple of 3, must n be a multiple of 3?

6. Show that, for every natural number n, the number $2^{n+2} + 3^{2n+1}$ is a multiple of 7.

7. Write down carefully (while not looking at your notes) a proof that there are infinitely many primes. By considering numbers of the form $4p_1p_2 \ldots p_k - 1$, prove that there are infinitely many primes of the form $4n-1$. What would go wrong if we tried a similar proof to show that there are infinitely many primes of the form $4n+1$?

8. Prove that $2^{2^n} - 1$ has at least n distinct prime factors.

9. We are given an operation $*$ on the natural numbers, satisfying
(i) $1 * n = n + 1$ for all n
(ii) $m * 1 = (m-1) * 2$ for all $m > 1$
(iii) $m * n = (m-1) * (m * (n-1))$ for all $m, n > 1$.
Find the value of $5 * 5$.

10. Suppose that we have some natural numbers (not necessarily distinct) whose sum is 100. How large can their product be?

11. We define the *unnatural numbers* to be the natural numbers whose last digit is 1; that is, the numbers 1, 11, 21, 31, ... We say that an unnatural number $n > 1$ is *primal* if it is not divisible by any unnatural numbers other than itself and 1. Show that every unnatural number is a product of primal numbers. Is the factorization of an unnatural number into primal numbers unique (up to reordering of the factors)?

12. Let x, y and z be natural numbers satisfying $x^2 + y^2 + 1 = xyz$. Prove that $z = 3$.

$^+$13. Each of n elderly dons knows a piece of gossip not known to any of the others. They communicate by telephone, and in each call the two dons concerned reveal to each other all the information they know so far. What is the smallest number of calls that can be made in such a way that, at the end, all the dons know all the gossip?

THIS IS THE BACK OF THE SHEET. IF YOU HAVEN'T READ THE FRONT THEN TURN OVER NOW AND COME BACK LATER.

This material is not really contained within the course and is provided for interest. When you are supervised, you will probably want to prioritize talking about the material on the front.

0a) Write down a definition of multiplication for natural numbers. Prove that multiplication is distributive over addition: that is, for all natural numbers a, b and c we have $a(b+c) = ab + ac$.

[Hint: first define $a1$ *for all natural numbers* a*. Then, assuming that you have defined* ab *for all natural numbers* a*, define* $a(b+1)$ *for all natural numbers* a*. For the proof, you may find it useful to use the fact proved in lectures that addition is associative.]*

b) Write down a definition of multiplication for rational numbers. Show that multiplication of rational numbers is well-defined.

c) Let x be a non-zero rational number. Show that there is a rational number y such that $xy = 1$.

d) Show that if a and b are natural numbers such that a is a factor of b then $a \leqslant b$.

e) If you want, think about how you might construct the integers from the natural numbers. How should we define addition, and what do we need to prove to show that the definition makes sense?