Numbers and Sets (2015–16) Example Sheet 2 of 4

1. Show that a number is divisible by 9 if, and only if, the sum of its digits is divisible by 9.

2. The *Fibonacci numbers* $F_0, F_1, F_2 \ldots$ are defined by $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$. Is F_{2015} even or odd? Is it a multiple of 3?

Show (by induction on k or otherwise) that $F_{n+k} = F_k F_{n+1} + F_{k-1} F_n$ for $k \geq 1$. Deduce that $(F_m, F_n) = (F_{m-n}, F_n)$, and thence that $(F_m, F_n) = F_{(m,n)}$.

3. Solve (i.e., find all solutions of) these congruences:-

(i) $77x \equiv 11 \pmod{40}$, (ii) $12y \equiv 30 \pmod{54}$,

(iii) $z \equiv 13 \pmod{21}$ and $3z \equiv 2 \pmod{17}$ simultaneously.

4. Do there exist 100 consecutive natural numbers, each of which has a proper square factor?

5. Without using a calculator, evaluate $20!21^{20} \pmod{23}$ and $17^{10000} \pmod{30}$.

6. By considering the n fractions $\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n}$, or otherwise, prove that $n = \sum_{d|n} \varphi(d)$.

7. An RSA encryption scheme (n, e) has modulus $n = 187$ and encoding exponent $e = 7$. Find a suitable decoding exponent d. Check your answer by encoding the number 35 and then decoding the result. *(Remember, no calculators!)*

8. Let p be a prime of the form $3k + 2$. Show that if $x^3 \equiv 1 \pmod{p}$ then $x \equiv 1 \pmod{p}$. Deduce that every number is a cube (mod p): i.e., $y^3 \equiv a \pmod{p}$ is soluble for all $a \in \mathbb{Z}$. Is the same ever true if p is of the form $3k + 1$?

9. Let A be a set of n positive integers. Show that every sequence of 2^n numbers taken from A contains a consecutive block of numbers whose product is a square. (For instance, 2,5,3,2,5,2,3,5 contains the block 5,3,2,5,2,3.)

10. Use the inclusion-exclusion principle to count the number of primes less than 121.

11. How many subsets of $\{1, 2, \ldots, n\}$ are there of even size?

12. By suitably interpreting each side, or otherwise, establish the identities

$$\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n-1}{k} + \binom{n}{k} = \binom{n+1}{k+1}$$

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$$

13. Show that $a^3 + b^5 = 7^{7^{7^7}}$ has no solution with $a, b \in \mathbb{Z}$.

14. Construct a function $f : \mathbb{R} \to \mathbb{R}$ that takes every value on every interval — in other words, for every $a < b$ and every c there is an x with $a < x < b$ such that $f(x) = c$.