Numbers and Sets (2015–16) Example Sheet 1 of 4

1. For which $n \in \mathbb{N}$, if any, are three numbers of the form $n, n+2, n+4$ all prime?

2. Between 0 and 10 there are four primes. Another example of two consecutive multiples of ten, between which there are four primes, is 10 and 20. Are there further examples?

3. If n^2 is a multiple of 3, must n be a multiple of 3?

4. Write down the negations of the following assertions (where $m, n, a, b \in \mathbb{N}$):

(i) if Coke is not worse than Pepsi then Corbyn can be King

(ii) $\forall m \, \exists n \, \forall a \, \forall b \;\; (n \geq m) \;\wedge\; [(a = 1) \vee (b = 1) \vee (ab \neq n)]$

Is (ii) true?

5. The sum of some (not necessarily distinct) natural numbers is 100. How large can their product be?

6. Do there exist 100 consecutive natural numbers none of which is prime?

7. In the sequence 41, 43, 47, 53, 61, ..., each difference is two more than the previous one. Are all the numbers in the sequence prime?

8. Find the highest common factor of 12345 and 54321. Find $u, v \in \mathbb{Z}$ with $76u + 45v = 1$. Does $3381x + 2646y = 21$ have an integer solution?

9. Find the convergents to the fraction $\frac{57}{44}$. Then prove that if x and y are integers such that $57x + 44y = 1$, then $x = 17 - 44k$ and $y = 57k - 22$ for some $k \in \mathbb{Z}$.

10. Let $a, b, c, d \in \mathbb{N}$. Must the numbers $(a, b)(c, d)$ and (ac, bd) be equal? If not, must one be a factor of the other? If $(a, b) = (a, c) = 1$, must we have $(a, bc) = 1$?

11. Show that, for any a, $b \in \mathbb{N}$, the number $\ell = ab/(a, b)$ is an integer (called the *least common multiple* of a and b). Show also that ℓ is divisible by both a and b, and that if $n \in \mathbb{N}$ is divisible by both a and b then $\ell \mid n$.

[Give two proofs, one based on uniqueness of prime factorization, the other Bezout-like.]

12. The *repeat* of a natural number is obtained by writing it twice in a row (for example, the repeat of 356 is 356356). Is there a number whose repeat is a perfect square?

13. Show that the exponent of the prime p in the prime factorisation of $n!$ is $\sum_{i \geq 1} \lfloor n/p^i \rfloor$, where $\lfloor x \rfloor$ denotes the integer part of x. Prove that this equals $(n - S_n)/(p - 1)$, where S_n is the sum of the digits in the base p representation of n. Hence show that $10^{249} \mid 1000!$.

14. Let $a < b$ be distinct natural numbers. Prove that every block of b consecutive natural numbers contains two distinct numbers whose product is divisible by ab. Suppose now $a < b < c$. Must every block of c consecutive numbers contain three distinct numbers whose product is divisible by abc?