1. Show that a number is divisible by 9 if, and only if, the sum of its digits is divisible by 9.

2. The *Fibonacci numbers* $F_0, F_1, F_2 \dots$ are defined by $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$. Is F_{2010} even or odd? Is it a multiple of 3?

Show (by induction on k or otherwise) that $F_{n+k} = F_k F_{n+1} + F_{k-1} F_n$ for $k \geq 1$. Deduce that $(F_m, F_n) = (F_{m-n}, F_n)$, and thence that $(F_m, F_n) = F_{(m,n)}$.

3. Let p be prime. Prove that if $0 < k < p$ then $\binom{p}{k} \equiv 0 \pmod{p}$. If you do this by using a formula for $\binom{p}{k}$ then argue correctly. Can you give a proof directly from the definition?

4. Solve these congruences:-

(i) $77x \equiv 11 \pmod{40}$, (ii) $12y \equiv 30 \pmod{54}$,

(iii) $z \equiv 13 \pmod{21}$ and $3z \equiv 2 \pmod{17}$ simultaneously.

5. Do there exist 100 consecutive natural numbers, each of which has a proper square factor?

6. Show that the exponent of the prime p in the prime factorisation of $n!$ is $\sum_{i \geq 1} \lfloor n/p^i \rfloor$, where $\lfloor x \rfloor$ denotes the integer part of x. Prove that this equals $(n - S_n)/(p-1)$, where S_n is the sum of the digits in the base p representation of n. Evaluate $1000! \pmod{10^{249}}$.

7. Without using a calculator, evaluate $20!21^{20} \pmod{23}$ and $17^{10000} \pmod{30}$.

8. By considering the n fractions $\frac{1}{n}, \frac{2}{n}, \dots, \frac{n}{n}$, or otherwise, prove that $n = \sum_{d|n} \varphi(d)$.

9. An RSA encryption scheme (n, e) has modulus $n = 187$ and encoding exponent $e = 7$. Find a suitable decoding exponent d. Check your answer by encoding the number 35 and then decoding the result. *(Remember, no calculators!)*

10. Let p be a prime of the form $3k + 2$. Show that if $x^3 \equiv 1 \pmod{p}$ then $x \equiv 1 \pmod{p}$. Deduce that every number is a cube (mod p): i.e., $y^3 \equiv a \pmod{p}$ is soluble for all $a \in \mathbb{Z}$. Is the same ever true if p is of the form $3k + 1$?

11. Using the least upper bound axiom, prove that there is a real number x satisfying $x^3 = 2$.

12. Prove that $\sqrt{2} + \sqrt{3}$ is irrational and algebraic. Do the same for $2^{1/3} + 2^{2/3}$.

13. Suppose that $x \in \mathbb{R}$ and $x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \dots + a_0 = 0$, where $a_{n-1}, \dots, a_0 \in \mathbb{Z}$. Prove that either x is an integer or it is irrational.

14. Show that $\sqrt[100]{\sqrt{3} + \sqrt{2}} + \sqrt[100]{\sqrt{3} - \sqrt{2}}$ is irrational.

15. Show that $a^4 + b^7 = 11^{11}$ has no solution with $a, b \in \mathbb{Z}$.