Numbers and Sets (2014–15) Example Sheet 1 of 4

1. For which $n \in \mathbb{N}$, if any, are three numbers of the form $n, n+2, n+4$ all prime?

2. Between 0 and 10 there are four primes. Another example of two consecutive multiples of ten, between which there are four primes, is 10 and 20. Are there further examples?

3. If n^2 is a multiple of 3, must n be a multiple of 3?

4. Write down the negations of the following assertions (where $m, n, a, b \in \mathbb{N}$):

(i) if Coke is not worse than Pepsi then Osborne hasn't a clue what he's about.

(ii) $\forall m \, \exists n \, \forall a \, \forall b \;\; (n \geq m) \;\wedge\; [(a = 1) \vee (b = 1) \vee (ab \neq n)]$,

5. The sum of some (not necessarily distinct) natural numbers is 100. How large can their product be?

6. Prove that $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.

7. The *symmetric difference* $A \triangle B$ of two sets A and B is the set of elements that belong to exactly one of A and B. Express this in terms of $\cup$, $\cap$ and $\setminus$. Prove that $\triangle$ is associative.

8. Let $A_1, A_2, A_3, \ldots$ be sets such that $A_1 \cap A_2 \cap \ldots \cap A_n \neq \emptyset$ holds for all n. Must it be that $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$?

9. Prove that $f \circ g$ is injective if f and g are injective. Does $f \circ g$ injective imply f injective? Does it imply g injective? What if we replace 'injective' by 'surjective' *passim*?

10. Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4, 5\}$? How many functions $A \to B$ are there? How many are injections? Count the number of surjections $B \to A$.

11. Let $f : X \to Y$ and let $C, D \subset Y$. Prove that $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$. Let $A, B \subset X$. Must it be true that $f(A \cap B) = f(A) \cap f(B)$?

12. Define a relation R on $\mathbb{N}$ by setting aRb if $a \mid b$ or $b \mid a$. Is R an equivalence relation?

13. The relation S *contains* the relation R if aSb whenever aRb. Let R be the relation on $\mathbb{Z}$ 'aRb if $b = a + 3$'. How many equivalence relations on $\mathbb{Z}$ contain R?

14. Construct a function $f : \mathbb{R} \to \mathbb{R}$ that takes every value on every interval — in other words, for every $a < b$ and every c there is an x with $a < x < b$ such that $f(x) = c$.

15. Find a bijection $f : \mathbb{Q} \to \mathbb{Q} \setminus \{0\}$. Can f be strictly increasing (that is, $f(x) < f(y)$ whenever $x < y$)?