1. Find the highest common factor of 12345 and 54321.

2. Find integers x and y with $76x + 45y = 1$. Do there exist integers x and y with $3381x + 2646y = 21$?

3. Prove that if a is coprime to b and also to c then it is coprime to bc. Give two proofs: one based on Euclid's algorithm / Bezout's theorem and one based on prime factorisation.

4. Is it true that for all positive integers a, b, c, d we have $(a, b)(c, d) = (ac, bd)$?

5. Show that a positive integer n is a multiple of 9 if and only if the sum of its digits is a multiple of 9.

6. The *Fibonacci numbers* $F_1, F_2, F_3, \ldots$ are defined by: $F_1 = F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all $n > 2$ (so eg. $F_3 = 2$, $F_4 = 3$, $F_5 = 5$). Is F_{2013} even or odd? Is it a multiple of 3?

7. Solve (ie. find all solutions of) the equations
(i) $7x \equiv 77$ (40)
(ii) $12y \equiv 30$ (54)
(iii) $3z \equiv 2$ (17) and $4z \equiv 3$ (19).

8. An RSA encryption scheme (n, e) has modulus $n = 187$ and coding exponent $e = 7$. By prime-factorising n, find a suitable decoding exponent d. Check your answer (without electronic assistance) by encoding the number 35 and then decoding the result.

9. Explain (without electronic assistance) why 23 cannot divide $10^{881} - 1$.

10. Let p be a prime of the form $3k + 2$. Show that, in $\mathbb{Z}_p$, the only solution to $x^3 = 1$ is $x = 1$. Deduce, or prove directly, that every element of $\mathbb{Z}_p$ has a cube root.

11. By considering numbers of the form $(2p_1p_2 \ldots p_k)^2 + 1$, prove that there are infinitely many primes of the form $4n + 1$.

12. What is the 5th-last digit of $5^{5^{5^{5^{5}}}}$?

13. Show that 19^{19} is not the sum of a fourth power and a (positive or negative) cube.

14. Let a and b be distinct positive integers, with say $a < b$. Prove that every block of b consecutive positive integers contains two distinct numbers whose product is a multiple of ab. If a, b and c are distinct positive integers, with say $a < b < c$, must every block of c consecutive positive integers contains three distinct numbers whose product is a multiple of abc?

$^{+}$15. Let n and k be positive integers. Suppose that n is a kth power (mod p) for all primes p. Must n be a kth power?