Mich. 2012 **NUMBERS AND SETS – EXAMPLES 2** IBL

1. Find the highest common factor of 12345 and 54321.

2. Find integers x and y with $76x + 45y = 1$. Do there exist integers x and y with $3528x + 966y = 24$?

3. Prove that if a is coprime to b and also to c then it is coprime to bc. Give two proofs: one based on Euclid's algorithm / Bezout's theorem and one based on prime factorisation.

4. Is it true that for all positive integers a, b, c, d we have $(a, b)(c, d) = (ac, bd)$?

5. Show that a positive integer n is a multiple of 9 if and only if the sum of its digits is a multiple of 9.

6. The *Fibonacci numbers* $F_1, F_2, F_3, \ldots$ are defined by: $F_1 = F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all $n > 2$ (so eg. $F_3 = 2$, $F_4 = 3$, $F_5 = 5$). Is F_{2012} even or odd? Is it a multiple of 3?

7. Solve (ie. find all solutions of) the equations
(i) $7x \equiv 77\ (40)$
(ii) $12y \equiv 30\ (54)$
(iii) $3z \equiv 2\ (17)$ and $4z \equiv 3\ (19)$.

8. An RSA encryption scheme (n, e) has modulus $n = 187$ and coding exponent $e = 7$. By prime-factorising n, find a suitable decoding exponent d. If you have a calculator, check your answer by encoding the number 35 and then decoding the result.

9. Explain (without using a calculator) why 23 cannot divide $10^{881} - 1$.

10. Let p be a prime of the form $3k + 2$. Show that, in $\mathbb{Z}_p$, the only solution to $x^3 = 1$ is $x = 1$. Deduce, or prove directly, that every element of $\mathbb{Z}_p$ has a cube root.

11. By considering numbers of the form $(2p_1p_2 \ldots p_k)^2 + 1$, prove that there are infinitely many primes of the form $4n + 1$.

12. Let a and b be distinct positive integers, with say $a < b$. Prove that every block of b consecutive positive integers contains two distinct numbers whose product is a multiple of ab. If a, b and c are distinct positive integers, with say $a < b < c$, must every block of c consecutive positive integers contains three distinct numbers whose product is a multiple of abc?

13. Is there a positive integer n for which $n^7 - 77$ is a Fibonacci number?

$^+$14. Let n be a fixed positive integer. Show that, for sufficiently large prime p (ie. for all but finitely many primes p), the equation $x^n + y^n = z^n$ has a solution in $\mathbb{Z}_p$ with $x, y, z \neq 0$.