# Numbers and Sets (2011–12) Example Sheet 3 of 4

**1.** Show that a number is divisible by 9 if, and only if, the sum of its digits is divisible by 9.

**2.** The *Fibonacci numbers* $F_0, F_1, F_2 \ldots$ are defined by $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$. Is $F_{2010}$ even or odd? Is it a multiple of $3$?

Show (by induction on $k$ or otherwise) that $F_{n+k} = F_k F_{n+1} + F_{k-1} F_n$ for $k \geq 1$. Deduce that $(F_m, F_n) = (F_{m-n}, F_n)$, and thence that $(F_m, F_n) = F_{(m,n)}$.

**3.** Let $p$ be prime. Prove that if $0 < k < p$ then $\binom{p}{k} \equiv 0 \pmod{p}$. If you do this by using a formula for $\binom{p}{k}$ then argue correctly. Can you give a proof directly from the definition?

**4.** Solve these congruences:-

(i) $77x \equiv 11 \pmod{40}$, (ii) $12y \equiv 30 \pmod{54}$,

(iii) $z \equiv 13 \pmod{21}$ and $3z \equiv 2 \pmod{17}$.

**5.** Do there exist 100 consecutive natural numbers, each of which has a proper square factor?

**6.** Show that the exponent of the prime $p$ in the prime factorisation of $n!$ is $\sum_{i\geq 1} \lfloor n/p^i \rfloor$, where $\lfloor x \rfloor$ denotes the integer part of $x$. Prove that this equals $(n - S_n)/(p-1)$, where $S_n$ is the sum of the digits in the base $p$ representation of $n$. Evaluate $1000! \pmod{10^{249}}$.

**7.** Without using a calculator, evaluate $20!21^{20} \pmod{23}$ and $17^{10000} \pmod{30}$.

**8.** By considering the $n$ fractions $\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n}$, or otherwise, prove that $n = \sum_{d|n} \varphi(d)$.

**9.** An RSA encryption scheme $(n, e)$ has modulus $n = 187$ and encoding exponent $e = 7$. Find a suitable decoding exponent $d$. Check your answer by encoding the number 35 and then decoding the result.

**10.** Let $p$ be a prime of the form $3k + 2$. Show that if $x^3 \equiv 1$ (mod $p$) then $x \equiv 1$ (mod $p$). Deduce that every number is a cube (mod $p$): i.e., $y^3 \equiv a$ (mod $p$) is soluble for all $a \in \mathbb{Z}$. Is the same true if $p$ is of the form $3k + 1$?

**11.** Using the least upper bound axiom, prove that there is a real number $x$ satisfying $x^3 = 2$.

**12.** Prove that $\sqrt{2} + \sqrt{3}$ is irrational and algebraic. Do the same for $2^{1/3} + 2^{2/3}$.

**13.** Suppose that $x \in \mathbb{R}$ and $x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_0 = 0$, where $a_{n-1}, \ldots, a_0 \in \mathbb{Z}$. Prove that either $x$ is an integer or it is irrational.

**14.** Show that $\sqrt[100]{\sqrt{3} + \sqrt{2}} + \sqrt[100]{\sqrt{3} - \sqrt{2}}$ is irrational.

**15.** Show that $a^4 + b^7 = 11^{11}$ has no solution with $a, b \in \mathbb{Z}$.