Numbers and Sets (2009–10) Example Sheet 4 of 4

1. Define a sequence $(x_n)_{n=1}^{\infty}$ by setting $x_1 = 1$ and $x_{n+1} = \frac{x_n}{1+\sqrt{x_n}}$ for all $n \geq 1$. Show that $(x_n)_{n=1}^{\infty}$ converges, and determine its limit.

2. Let $(a_n)_{n=1}^{\infty}$ be a sequence of reals. Show that if $(a_n)_{n=1}^{\infty}$ is convergent then we must have $a_n - a_{n-1} \to 0$. If $a_n - a_{n-1} \to 0$, must $(a_n)_{n=1}^{\infty}$ be convergent?

3. Let $[a_n, b_n]$, $n = 1, 2, \ldots$, be closed intervals with $[a_n, b_n] \cap [a_m, b_m] \neq \emptyset$ for all n, m. Prove that $\bigcap_{n=1}^{\infty} [a_n, b_n] \neq \emptyset$.

4. Which of the following sequences $(x_n)_{n=1}^{\infty}$ converge?

$$x_n = \frac{3n}{n+3} \qquad x_n = \frac{n^{100}}{2^n} \qquad x_n = \sqrt{n+1} - \sqrt{n} \qquad x_n = (n!)^{1/n}$$

5. Which of the following series converge?

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2} \qquad \sum_{n=1}^{\infty} \frac{n!}{n^n} \qquad \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+n}}$$

6. Let $a_n \in \mathbb{R}$ and let $b_n = \frac{1}{n} \sum_{i=1}^{n} a_i$. Show that, if $a_n \to a$ as $n \to \infty$, then $b_n \to a$ also.

7. The series $\sum_{n=1}^{\infty} x_n$ converges. Must $\sum_{n=1}^{\infty} x_n/n$ also converge?

8. A real number $r = 0 \cdot d_1 d_2 d_3 \ldots$ is called *repetitive* if its decimal expansion contains arbitrarily long blocks that are the same; that is, for every k there exist distinct m and n such that $d_m = d_n,\ d_{m+1} = d_{n+1},\ \ldots,\ d_{m+k} = d_{n+k}$. Prove that the square of a repetitive number is repetitive.

9. Show that any collection of pairwise disjoint discs in the plane is countable. What happens if we replace 'discs' by 'circles'?

10. Show that the collection of all finite subsets of $\mathbb{N}$ is countable. What goes wrong if we try to use the diagonal argument to show that it is uncountable?

11. A function $f : \mathbb{N} \to \mathbb{N}$ is *increasing* if $f(n+1) \geq f(n)$ for all n and *decreasing* if $f(n+1) \leq f(n)$ for all n. Is the set of increasing functions countable or uncountable? What about the set of decreasing functions?

12. Find an injection $\mathbb{R}^2 \to \mathbb{R}$. Is there an injection from the set of all real sequences to $\mathbb{R}$?

13. Is there an uncountable family $S \subset \mathcal{P}\mathbb{N}$ such that $A \cap B$ is finite for all distinct $A, B \in S$?

14. For each $x \in \mathbb{R}$ we are given an interval $I_x = [x - \delta_x, x + \delta_x]$ with $\delta_x \geq 0$. Moreover, for each $x, y \in \mathbb{R}$ with $y \in I_x$, we have $\delta_y < \delta_x$. Show that $\delta_x = 0$ for uncountably many x.