Numbers and Sets (2009–10) Example Sheet 2 of 4

1. Prove that $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.

2. The *symmetric difference* $A \triangle B$ of two sets A and B is the set of elements that belong to exactly one of A and B. Express this in terms of $\cup$, $\cap$ and $\setminus$. Prove that $\triangle$ is associative.

3. Let $A_1, A_2, A_3, \ldots$ be sets such that $A_1 \cap A_2 \cap \ldots \cap A_n \neq \emptyset$ holds for all n. Must it be that $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$?

4. Prove that $f \circ g$ is injective if f and g are injective. Does $f \circ g$ injective imply f injective? Does it imply g injective? What if we replace 'injective' by 'surjective' *passim*?

5. Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4, 5\}$? How many functions $A \to B$ are there? How many are injections? Count the number of surjections $B \to A$.

6. Let $f : X \to Y$ and let $C, D \subset Y$. Prove that $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$. Let $A, B \subset X$. Must $f(A \cap B) = f(A) \cap f(B)$ be true?

7. The relation S *contains* the relation R if aSb whenever aRb. Let R be the relation on $\mathbb{Z}$ 'aRb if $b = a + 3$'. How many equivalence relations on $\mathbb{Z}$ contain R?

8. Let A be a set of n positive integers. Show that every sequence of 2^n numbers taken from A contains a consecutive block of numbers whose product is a square. (For instance, 2,5,3,2,5,2,3,5 contains the block 5,3,2,5,2,3.)

9. Use the inclusion-exclusion principle to count the number of primes less than 121.

10. How many subsets of $\{1, 2, \ldots, n\}$ are there of even size?

11. By suitably interpreting each side, or otherwise, establish the identities

$$\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}$$

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$$

12. A *triomino* is an L-shaped pattern made from three square tiles. A $2^k \times 2^k$ chessboard, whose squares are the same size as the tiles, has one of its squares painted puce. Show that the chessboard can be covered with triominoes so that only the puce square is exposed.

13. Evaluate $a(4, 4)$ for the function $a(m, n)$, which is defined for integers $m, n \geq 0$ by

$$\begin{aligned} a(0, n) &= n + 1, \text{ if } n \geq 0; \\ a(m, 0) &= a(m-1, 1), \text{ if } m > 0; \\ a(m, n) &= a(m-1, a(m, n-1)), \text{ if } m > 0, \text{ and } n > 0\,. \end{aligned}$$

14. Find a bijection $f : \mathbb{Q} \to \mathbb{Q} \setminus \{0\}$. Can f be strictly increasing (that is, $f(x) < f(y)$ whenever $x < y$)?