Numbers and Sets (2009–10) Example Sheet 1 of 4

1. Write down the negations of the following assertions (where $m, n, a, b \in \mathbb{N}$):

(i) if Coke is not worse than Pepsi then nothing Mandelson says can be trusted.

(ii) $\forall m \, \exists n \, \forall a \, \forall b \;\; (n \geq m) \;\wedge\; [(a = 1) \vee (b = 1) \vee (ab \neq n)]$,

2. For which $n \in \mathbb{N}$, if any, are three numbers of the form $n, n+2, n+4$ all prime?

3. Between 0 and 10 there are four primes. The same is true between 10 and 20. Does it ever happen again between two consecutive multiples of ten?

4. If n^2 is a multiple of 3, must n be a multiple of 3?

5. The sum of some (not necessarily distinct) natural numbers is 100. How large can their product be?

6. Find the highest common factor of 12345 and 54321.

7. Find integers u and v with $76u + 45v = 1$. Does $3528x + 966y = 24$ have an integer solution?

8. Find the convergents to the fraction $\frac{57}{44}$. Prove that if x and y are integers such that $57x + 44y = 1$, then $x = 17 - 44k$ and $y = 57k - 22$ for some $k \in \mathbb{Z}$.

9. Which of these are true for all natural numbers a, b and c?

(i) $(a, b)\,(a, c) = (a^2, bc)$.

(ii) If $(a, b) = (a, c) = 1$ then $(a, bc) = 1$.

10. Show that, for any a, $b \in \mathbb{N}$, the number $\ell = ab/(a, b)$ is an integer (called the *least common multiple* of a and b). Show also that ℓ is divisible by both a and b, and that if $n \in \mathbb{N}$ is divisible by both a and b then $\ell \mid n$.

11. Do there exist 100 consecutive natural numbers none of which is prime?

12. In the sequence 41, 43, 47, 53, 61, ..., each difference is two more than the previous one. Are all the numbers in the sequence prime?

13. The *repeat* of a natural number is obtained by writing it twice in a row (for example, the repeat of 356 is 356356). Is there a number whose repeat is a perfect square?

14. Define $\mu : \mathbb{N} \to \mathbb{Z}$ by $\mu(n) = (-1)^k$ if n is a product of k distinct primes (including the case $k = 0$) and $\mu(n) = 0$ otherwise. Prove that $\sum_{d|n} \mu(d) = 0$ unless $n = 1$.