NUMBERS AND SETS — EXAMPLES 4

PTJ

Important Note These questions are (deliberately) not all equally difficult or equally long. In general, the first ten questions on each sheet are intended to be ones which can be tackled (perhaps with a hint or two from a supervisor) by anyone who has understood the relevant material in the lectures; questions from 11 onwards may be more challenging, and are intended for those who find the standard material easy.

1. Let $A_1, A_2, A_3, \ldots$ be a sequence of subsets of $\mathbf{R}$ such that $A_1 \cap A_2 \cap \cdots \cap A_n \neq \emptyset$ for each $n \geq 1$. Does it follow that $\bigcap_{n=1}^{\infty} A_n$ (that is, the intersection of all the A_n's) is nonempty? Does the answer change if you are given the extra information that each A_n is a *closed interval*, that is a set of the form $[a_n, b_n] = \{x \in \mathbf{R} \mid a_n \leq x \leq b_n\}$ for some pair (a_n, b_n) of real numbers with $a_n < b_n$?

2. The *Cantor set* C is defined to be $\bigcap_{n=1}^{\infty} A_n$, where A_1 is the closed interval $[0, 1]$ and, for each $n > 1$, A_n is obtained from A_{n-1} by deleting the middle third of each interval in A_{n-1} — thus $A_2 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, $A_3 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$, and so on. Show that a real number x belongs to C if and only if it has an expansion $\sum_{n=1}^{\infty} a_n/3^n$ in base 3, where each a_n is either 0 or 2. Deduce that C is uncountable.

3. A set A is said to be *Dedekind-infinite* if there exists an injective function $f \colon A \to A$ which is not surjective. Show that A is Dedekind-infinite if and only if there is an injective function $\mathbf{N} \to A$. [Hint: given f as in the definition, choose $a_0 \in A$ which is not in its image and consider the mapping $n \mapsto f^n(a_0)$.]

4. Construct an injection from $\mathbf{R} \times \mathbf{R}$ to $\mathbf{R}$.

5. Construct a bijection from $\mathbf{Q}$ to $\mathbf{Q} \setminus \{0\}$. Can we find such a bijection f which is order-preserving (that is, satisfies $f(x) < f(y)$ whenever $x < y$)?

6. Show that the set of all finite subsets of $\mathbf{N}$ is countable. What goes wrong if we try to use a diagonal argument to show that it is uncountable?

7. Show that there does not exist an uncountable family of pairwise disjoint discs in the plane. What happens if we replace 'discs' by 'circles'?

8. A function $f : \mathbf{N} \to \mathbf{N}$ is *increasing* if $f(n + 1) \geq f(n)$ for all n, and *decreasing* if $f(n + 1) \leq f(n)$ for all n. Is the set of increasing functions countable or uncountable? What about the set of decreasing functions?

9. Let $f : \mathbf{R} \to \mathbf{R}$ be a function such that $f(x) \geq 0$ for all x, and such that, whenever $y \neq x$ and $x - f(x) \leq y \leq x + f(x)$, we have $f(y) < f(x)$. Show that there are uncountably many x such that $f(x) = 0$.

10. Using the Cantor–Bernstein theorem, show that there is a bijection between the set of all subsets of $\mathbf{R}$ and the set of all functions $\mathbf{R} \to \mathbf{R}$. [Hint: recall that a function $f \colon \mathbf{R} \to \mathbf{R}$ may be represented by the set of ordered pairs $\{(x, f(x)) \mid x \in \mathbf{R}\}$.]

11. Let S be a collection of subsets of $\mathbf{N}$ such that for every $A, B \in S$ we have either $A \subseteq B$ or $B \subseteq A$. Can S be uncountable?

12. (a) By a *finite continued fraction* we mean an expression of the form $[a_0, a_1, a_2, \ldots, a_n]$ where $n \geq 0$ and the a_i are integers, all except possibly a_0 being strictly positive. We assign a real value to each such expression by induction on n, as follows:
if $n = 0$, $[a_0] = a_0$;
if $n > 0$, $[a_0, a_1, \ldots, a_n] = a_0 + 1/[a_1, \ldots, a_n]$.
Show that the value of any finite continued fraction is rational. Show also that any rational number can be represented in exactly two ways by a finite continued fraction. [Hint: given $x \in \mathbf{Q} \setminus \mathbf{Z}$, set $a_0 = \lfloor x \rfloor$ and observe that $1/(x - a_0)$ has smaller denominator than x.]

(b) An *infinite continued fraction* is an expression $[a_0, a_1, a_2, \ldots]$ where the a_i satisfy the same conditions as before (but there are infinitely many of them). Show that, for any such expression, the values of the finite continued fractions $[a_0, a_1, \ldots, a_n]$ converge to a limit as $n \to \infty$; we define this limit to be the value of the infinite continued fraction. Show also that any irrational number can be represented by a unique infinite continued fraction.

(c) A real number x is called a *quadratic irrational* if it is not rational, but it is a root of a quadratic polynomial with rational coefficients; that is, $ax^2 + bx + c = 0$ for some $a, b, c \in \mathbf{Q}$, not all zero. Show that if x is a quadratic irrational then so are $1/x$ and $a_0 + x$ for any $a_0 \in \mathbf{Z}$. Deduce that if the continued fraction representation of x is eventually periodic (i.e., there exist positive integers k and n_0 such that $a_{n+k} = a_n$ whenever $n \geq n_0$) then x is a quadratic irrational.

(d) Find the continued fraction representations of $\sqrt{2}$, $\sqrt{3}$ and the golden ratio τ (cf. question 11 on sheet 3).

(e) Assuming the result that *Pell's equation* $x^2 - dy^2 = 1$ always has a solution in integers x, y when d is an integer but not a perfect square, show that for any such d the continued fraction representation of $\sqrt{d}$ is eventually periodic. Deduce that the continued fraction representation of every quadratic irrational is eventually periodic.

13. Consider the function $f\colon \mathbf{R} \to \mathbf{R}$ defined as follows. We set $f(n) = n$ for each integer n. We then define f on the rationals by the following recursion: assume $f(x)$ has been defined for all rationals with denominators less than q; then if $x = p/q$ is a rational with denominator q (in lowest terms), we set $f(x) = \frac{1}{2}(f(x^-) + f(x^+))$, where x^- and x^+ are the rationals closest to x on either side with denominators less than q. [Thus $f(\frac{1}{2}) = \frac{1}{2}$, $f(\frac{1}{3}) = \frac{1}{4}$, $f(\frac{1}{4}) = \frac{1}{8}$, $f(\frac{2}{5}) = \frac{3}{8}$, and so on.] Finally, for any irrational x, we define $f(x) = \sup\,\{f(y) \mid y \in \mathbf{Q}, y < x\}$.

(a) Show that f maps $\mathbf{Q}$ bijectively to the set $\mathbf{D}$ of *dyadic rationals*, i.e. rationals whose denominators are powers of 2.

(b) Show that f coincides with the function defined as follows: suppose x has continued fraction representation $[a_0, a_1, a_2, a_3, \ldots]$; then $f(x)$ has integer part a_0, and the expansion of its fractional part in base 2 consists of a block of $(a_1 - 1)$ 0's, followed by a block of a_2 1's, followed by a_3 0's, and so on (ending with an infinite string of 0's or 1's, as appropriate, if the continued fraction is finite).

(c) Deduce that f maps the set of quadratic irrationals bijectively to $\mathbf{Q} \setminus \mathbf{D}$. Show in particular that $f(\tau) = \frac{5}{3}$.