Mich. 2007 **NUMBERS AND SETS — EXAMPLES 3** PTJ

Important Note These questions are (deliberately) not all equally difficult or equally long. In general, the first ten questions on each sheet are intended to be ones which can be tackled (perhaps with a hint or two from a supervisor) by anyone who has understood the relevant material in the lectures; questions from 11 onwards may be more challenging, and are intended for those who find the standard material easy.

1. Prove carefully, using the least upper bound axiom, that there is a real number x satisfying $x^3 = 2$. Prove also that such an x must be irrational.

2. Suppose that a real number x is a root of a monic integer polynomial; that is, we have an equation of the form $x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0$ with $a_1, a_2, \ldots, a_n \in \mathbf{Z}$. Prove that if x is not an integer then it must be irrational.

3. Show that $\sqrt{2} + \sqrt{3}$ is algebraic. Is it rational?

4. Prove that the usual order-relation $<$ is the only binary relation on $\mathbf{Q}$ which satisfies the axioms for an ordered field.

5. Let $\mathbf{Q}(\sqrt{2})$ denote the set of all real numbers of the form $a + b\sqrt{2}$, where a and b are rational. Show that $\mathbf{Q}(\sqrt{2})$ is a subfield of $\mathbf{R}$ (that is, it is closed under the operations of addition and multiplication, and of taking additive and multiplicative inverses). Show also that there are two different binary relations on $\mathbf{Q}(\sqrt{2})$ which make it into an ordered field.

6. Let (x_n) and (y_n) be sequences of real numbers. Prove that if $x_n \to 0$ and $y_n \to 0$ as $n \to \infty$, then $x_ny_n \to 0$. By considering $x_n - c$ and $y_n - d$, deduce that if $x_n \to c$ and $y_n \to d$ as $n \to \infty$, then $x_ny_n \to cd$.

7. Which of the following sequences (x_n) converge?

$$(a)\ x_n = \frac{3n}{n+3}\ ; \quad (b)\ x_n = n^{100}/2^n\ ; \quad (c)\ x_n = \sqrt{n+1} - \sqrt{n}\ ; \quad (d)\ x_n = (n!)^{1/n}\ .$$

8. Define a sequence (x_n) by setting $x_1 = 1$ and $x_{n+1} = x_n/(1 + \sqrt{x_n})$ for all $n \geq 1$. Show that (x_n) converges, and find its limit.

9. Which of the following series converge?

$$(a)\ \sum_{n=1}^{\infty} \frac{1}{1+n^2}\ ; \quad (b)\ \sum_{n=1}^{\infty} \frac{n!}{n^n}\ ; \quad (c)\ \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+n}}\ .$$

10. A real number $x = n \cdot a_1a_2a_3 \cdots$ is called *repetitive* if its decimal expansion contains pairs of arbitrarily long blocks which are the same, i.e. if for every k there exist distinct m and n such that $a_m = a_n,\ a_{m+1} = a_{n+1}, \ldots,\ a_{m+k} = a_{n+k}$. Prove that the square of a repetitive number is repetitive.

11. Let (F_n) denote the sequence of Fibonacci numbers (for the definition, see question 5 on sheet 2), and let $x_n = F_{n+1}/F_n$. Show that $x_n - x_{n+1} = (-1)^n/F_nF_{n+1}$ for all n; deduce that (x_n) converges to a limit τ as $n \to \infty$, and that τ is irrational and algebraic. [τ is traditionally called the *golden ratio*, even though it's irrational.] Show also that $F_n = (\tau^n - (-\tau)^{-n})/\sqrt{5}$ for all n.

12. (*a*) Let p_n denote the nth prime number (so $p_1 = 2, p_2 = 3, p_3 = 5, \ldots$), and let $u_n = (1 - \frac{1}{p_n})^{-1}$. Show that

$$\prod_{n=1}^{N} u_n \geq \sum_{k=1}^{p_{N+1}-1} k^{-1} ,$$

and deduce that the product on the left tends to infinity as $N \to \infty$.
(*b*) Assuming the result that $\sum_{n=1}^{\infty} x^n/n$ converges to $\log\ 1/(1-x)$ for $|x| < 1$ (which will — I hope — be proved in next term's Analysis course), show that $\log\ 1/(1-x) \leq x + x^2$ for $0 \leq x \leq \frac{1}{2}$.
(*c*) Deduce that $\sum_{n=1}^{\infty} p_n^{-1}$ diverges. [Thus, in some sense, there are more primes than perfect squares.]

13. Let (x_n) be a sequence of positive real numbers such that $\sum_{n=1}^{\infty} x_n$ is convergent. Show that there exists a sequence (y_n) such that $y_n/x_n \to \infty$ as $n \to \infty$, but $\sum_{n=1}^{\infty} y_n$ is still convergent.

14. [An example of a non-archimedean ordered field.] We define a *polynomial fraction* to be a formal expression of the form $(p(x)/q(x))$ where $p(x)$ and $q(x)$ are polynomials with real coefficients and $q(x)$ is not identically zero; a *rational function* is an equivalence class of polynomial fractions under the equivalence relation which identifies $(p(x)/q(x))$ with $(r(x)/s(x))$ if and only if $p(x)s(x) = q(x)r(x)$ for all x.
(*a*) Verify that the relation just described is indeed an equivalence relation.
(*b*) Define addition and multiplication of rational functions, and verify that they satisfy the usual laws of arithmetic, so that the set of all rational functions forms a field F. [This is just like the construction of $\mathbf{Q}$ from $\mathbf{Z}$ which we did in lectures; there is no need to write out all the proofs in detail, as long as you do enough to convince yourself — and your supervisor — that you *could* write them out if necessary.]
(*c*) We define a binary relation $<$ on F by setting $(p(x)/q(x)) < (r(x)/s(x))$ if and only if there exists $x_0 \in \mathbf{R}$ such that $q(x)r(x) - p(x)s(x)$ and $q(x)s(x)$ (are nonzero and) have the same sign for all $x > x_0$. Show that this definition makes F into an ordered field. [Hint: recall that a polynomial which is not identically zero has only finitely many roots.]
(*d*) Show that the fractions of the form $(a/1)$, where a is a real number regarded as a constant polynomial, form a subfield of F isomorphic to $\mathbf{R}$. Show also that this subset has an upper bound in F.
(*e*) If you're feeling really energetic, find all possible binary relations on F which make it into an ordered field. [Hint: once you know where $(x/1)$ lies in relation to the constant rational functions, you've determined the order completely.]