Mich. 2007 **NUMBERS AND SETS — EXAMPLES 1** PTJ

Important Note These questions are (deliberately) not all equally difficult or equally long. In general, the first ten questions on each sheet are intended to be ones which can be tackled (perhaps with a hint or two from a supervisor) by anyone who has understood the relevant material in the lectures; questions from 11 onwards may be more challenging, and are intended for those who find the standard material easy.

1. Express the following sentences in mathematical notation, using appropriate symbols to represent the properties and relationships involved. (For example, in (*a*) you could use $S(x, y)$ to denote 'x is a son of y', and in (*b*) you could use $R(d)$ to denote 'it is raining on day d'.)
(*a*) Any son of Joe is a friend of mine.
(*b*) Janet never cycles to work when it's raining.
(*c*) Every Prime Minister needs a Willie. (M. Thatcher)
(*d*) If Chelsea win the Premiership in 2008 then pigs will fly.
(*e*) There is always someone worse off than yourself.
(*f*) When everyone is somebody, then no-one's anybody. (W.S. Gilbert)

2. The *symmetric difference* $A \triangle B$ of two sets A and B is defined to be the set of elements that belong to one of A and B but not both. Express this definition in symbols. Show that an element belongs to $A \triangle (B \triangle C)$ if and only if it belongs to an odd number of the sets A, B and C, and deduce (or prove otherwise, if you prefer) that the operation $\triangle$ is associative — that is, we always have $A \triangle (B \triangle C) = (A \triangle B) \triangle C$. Show also that $(A \triangle B) \cap C = (A \cap C) \triangle (B \cap C)$. Do we also have $(A \triangle B) \cup C = (A \cup C) \triangle (B \cup C)$?

3. Let $f\colon A \to B$ and $g\colon B \to C$ be functions, and let $h = g \circ f\colon A \to C$.
(i) Show that if f and g are both injective, then h is injective.
(ii) Show that if h is injective, then f is injective, but g need not be.

Now formulate and prove similar results with the word 'injective' replaced by 'surjective'. [N.B.: I said the results were similar, not identical.]

4. (*a*) Is it possible for three consecutive odd positive integers to be prime?
(*b*) Suppose that four out of five consecutive odd integers, greater than 10, are prime. Show that they must be of the form $10n+1, 10n+3, 10n+7$ and $10n+9$ for some positive integer n, and further that n must be of the form $3k+1$ for some k. Observe that $11, 13, 17, 19$ is one such 'prime quadruplet'; can you find another?
(*c*) Can you find a block of 100 consecutive odd positive integers, none of which is prime?
(*d*) The numbers $41, 43, 47, 53$ and 61 are all prime. Does there ever exist a term in this sequence (where the $(n+1)$st term is obtained by adding $2n$ to the nth term) which is not prime?

5. Use the inclusion–exclusion principle to count the number of primes less than 169. [Hint: if a number less than n^2 is not prime, it has a prime factor less than n.]

6. Prove by induction that the following statements are true for all positive integers n:
(i) $2^{n+2} + 3^{2n+1}$ is a multiple of 7.
(ii) $1^3 + 3^3 + 5^3 + \cdots + (2n-1)^3 = n^2(2n^2 - 1)$.

7. You are given a $2^n \times 2^n$ grid of squares and a supply of L-shaped tiles each of which will exactly cover three squares on the grid. One square on the grid has been chosen and painted pink. Prove that it is possible to cover the rest of the grid with (non-overlapping) tiles, so that only the pink square is exposed.

8. Consider the following 'proofs by induction' that all positive integers are equal, and that all positive integers are interesting. Are they valid? If not, why not?
(*a*) We prove by induction on n that, for any n-element set $\{x_1, x_2, \ldots, x_n\}$ of positive integers, we have $x_1 = x_2 = \cdots = x_n$. Clearly, when $n = 1$ there is nothing to prove. So assume the result is true when $n = k$, and consider a $(k+1)$-element set $\{x_1, x_2, \ldots, x_k, x_{k+1}\}$. By the inductive hypothesis applied to the sets $\{x_1, x_2, \ldots, x_k\}$ and $\{x_2, x_3, \ldots, x_{k+1}\}$, we have $x_1 = x_2 = \cdots = x_k$ and $x_2 = x_3 = \cdots = x_{k+1}$. Hence $x_1 = x_2 = \cdots = x_{k+1}$, so the result is true for $n = k + 1$.
(*b*) Clearly, some positive integers are interesting: for example, 6 is the smallest perfect number, 1729 is the smallest number expressible as the sum of two cubes in two different ways, and so on. Suppose some positive integers were not interesting: then, by the well-ordering principle, there would be a smallest uninteresting number. But such a number would clearly be very interesting indeed. Therefore, all positive integers are interesting.

9. Give synthetic proofs (that is, proofs that involve constructing bijections between appropriate sets) of the following identities involving binomial coefficients:

(*a*)
$$\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1} .$$

(*b*)
$$\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n} .$$

[Hint for (*a*): consider the largest member of a $(k+1)$-element subset of $\{1, 2, \ldots, n+1\}$.]

10. A binary operation $*$, defined on the positive integers, satisfies the conditions
(*a*) $1 * n = n + 1$ for all n,
(*b*) $m * 1 = (m-1) * 2$ for all $m > 1$, and
(*c*) $m * n = (m-1) * (m * (n-1))$ for all $m, n > 1$.
Do these conditions suffice to determine $m * n$ uniquely for all m and n? If so, find the value of $5 * 5$.

11. A 'Chinese Rings' puzzle consists of a number of rings, which are linked together in a definite order, and a metal bar; each ring may be either on or off the bar at a given time. The linkage between the rings is such that the first ring may be moved on or off the bar at any time, but for $k > 1$ the kth ring may be moved on or off only when the $(k-1)$st ring is on and all earlier rings are off. Let $f(n)$ denote the number of moves required to get a puzzle with n rings from the 'all on' to the 'all off' position.
(*a*) Find an expression for $f(n)$ in terms of $f(n-1)$ and $f(n-2)$. [Hint: consider the sequences of moves before and after the nth ring is moved.]
(*b*) Use induction to prove that $f(n) = \lfloor 2^{n+1}/3 \rfloor$.
(*c*) Observe that in any possible position of the puzzle, with exactly two exceptions, there are two possible moves that you can make. Use this observation to establish the identity $f(n) + f(n-1) = 2^n - 1$, and hence obtain another proof of the formula in (*b*).

12. Find a positive integer k such that, for all positive integers n, the number $n^4 + k$ is composite.

13. The *repeat* of a positive integer is the number obtained by writing it twice in a row (so for example the repeat of 254 is 254254). Can you find a positive integer whose repeat is a perfect square?