

1. Prove carefully, using the least upper bound axiom, that there is a real number  $x$  satisfying  $x^3 = 2$ . Prove also that such an  $x$  must be irrational.
2. Prove that  $\sqrt{2} + \sqrt{3}$  is irrational and algebraic.
3. Suppose that the real number  $x$  is a root of a monic integer polynomial, ie. we have  $x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \dots + a_0 = 0$ , for some integers  $a_{n-1}, \dots, a_0$ . Prove that  $x$  is either integer or irrational.
4. Let  $(x_n)_{n=1}^{\infty}$  and  $(y_n)_{n=1}^{\infty}$  be sequences of reals. Show that if  $x_n \rightarrow 0$  and  $x_n \rightarrow 0$  then  $x_n y_n \rightarrow 0$ . By considering  $x_n - c$  and  $y_n - d$ , prove carefully that if  $x_n \rightarrow c$  and  $y_n \rightarrow d$  then  $x_n y_n \rightarrow cd$ .
5. Let  $(x_n)_{n=1}^{\infty}$  be a sequence of reals. Show that if  $(x_n)_{n=1}^{\infty}$  is convergent then we must have  $x_n - x_{n-1} \rightarrow 0$ . If  $x_n - x_{n-1} \rightarrow 0$ , must  $(x_n)_{n=1}^{\infty}$  be convergent?
6. Which of the following sequences  $(x_n)_{n=1}^{\infty}$  converge?
  - (i)  $x_n = \frac{3n}{n+3}$  ,   (ii)  $x_n = \frac{n^{100}}{2^n}$  ,   (iii)  $x_n = \sqrt{n+1} - \sqrt{n}$  ,   (iv)  $x_n = (n!)^{1/n}$  .
7. Which of the following series converge?
  - (i)  $\sum_{n=1}^{\infty} \frac{1}{1+n^2}$  ,   (ii)  $\sum_{n=1}^{\infty} \frac{n!}{n^n}$  ,   (iii)  $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+n}}$  .
8. Define a sequence  $(x_n)_{n=1}^{\infty}$  by setting  $x_1 = 1$  and  $x_{n+1} = \frac{x_n}{1+\sqrt{x_n}}$  for all  $n \geq 1$ . Show that  $(x_n)_{n=1}^{\infty}$  converges, and determine its limit.
9. A real number  $x = 0 \cdot x_1 x_2 x_3 \dots$  is called *repetitive* if its decimal expansion contains arbitrarily long blocks that are the same, ie. if for every  $k$  there exist distinct  $m$  and  $n$  such that  $x_m = x_n$ ,  $x_{m+1} = x_{n+1}$ ,  $\dots$ ,  $x_{m+k} = x_{n+k}$ . Prove that the square of a repetitive number is repetitive.
10. Show that if  $\sum_{n=1}^{\infty} x_n$  is a convergent series of reals, with all  $x_n$  positive, then  $\sum_{n=1}^{\infty} x_n^2$  is also convergent. What happens if we do not insist that the  $x_n$  are positive?
11. If  $\sum_{n=1}^{\infty} x_n$  is a convergent series of reals, must  $\sum_{n=1}^{\infty} x_n^3$  be convergent?
12. Show that  ${}^{100}\sqrt{\sqrt{3} + \sqrt{2}} + {}^{100}\sqrt{\sqrt{3} - \sqrt{2}}$  is irrational.
13. Let  $(x_n)_{n=1}^{\infty}$  be a real sequence with  $x_n \rightarrow 0$ . Prove carefully that we may choose  $(\epsilon_n)_{n=1}^{\infty}$ , with each  $\epsilon_n = \pm 1$ , such that  $\sum_{n=1}^{\infty} \epsilon_n x_n$  is convergent. If  $(y_n)_{n=1}^{\infty}$  is another real sequence tending to 0, can we choose the  $\epsilon_n$  so that  $\sum_{n=1}^{\infty} \epsilon_n y_n$  is convergent as well?
- +14. Let  $S$  be a (possibly infinite) set of odd positive integers. Prove that there exists a real sequence  $(x_n)_{n=1}^{\infty}$  such that, for each positive integer  $k$ , the series  $\sum_{n=1}^{\infty} x_n^k$  converges when  $k$  belongs to  $S$  and diverges when  $k$  does not belong to  $S$ .