IA Groups - Example Sheet 3

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1. Throughout this question 'identify' means 'to find a standard group that it is isomorphic to'.
   (a) Let $H \leqslant C_n$. Identify the quotient C_n/H.
   (b) Show that $N = \{e, (12)(34), (13)(24), (14)(23)\}$ is a normal subgroup of S_4. Identify the quotient S_4/N.
   (c) Show that any subgroup $N \leqslant D_{2n}$ consisting only of rotations is normal. Identify the quotient D_{2n}/N.
   (d) Given a group G, let G^2 denote the direct product $G \times G$. Consider the subgroup $\mathbb{Z}^2$ of the group $\mathbb{R}^2$. Identify the quotient $\mathbb{R}^2/\mathbb{Z}^2$.

2. Given subgroups H and N of a group G, show that $HN = \{hn : h \in H, n \in N\}$ is a subgroup of G if N is normal in G. If H and N are both finite, prove that $|HN| = \frac{|H| \cdot |N|}{|H \cap N|}$.

3. (a) Let H be a subgroup of a group G. Show that H is normal in G if and only if H is a union of some conjugacy classes of G.
   (b) Let N be a normal subgroup of index m in G. Show that $g^m \in N$ for any $g \in G$.

4. Let G be a group acting on a set X. If for $x, y \in X$, there is a $g \in G$ such that $g(x) = y$, show that $\mathrm{Stab}(y) = g \,\mathrm{Stab}(x) g^{-1}$.

5. Let G be a finite abelian group acting faithfully on a finite set X. Show that if the action is transitive, then $|G| = |X|$.

6. Show that D_{2n} has one conjugacy class of reflections if n is odd, and two conjugacy classes of reflections if n is even.

7. Let G be a finite group. Show that $g(H) = gHg^{-1}$ defines an action of G on the set of subgroups of G. Show that for $H \leqslant G$, the size of the orbit of H under this action is at most $|G : H|$. Deduce that if $H \neq G$, then G is not the union of all conjugates of H.

8. (a) Let G be a finite group and let H be a subgroup of index $k \neq 1$ in G. Suppose that $|G|$ does not divide $k!$. By considering the action of G on the set of left cosets of H in G, show that H contains a non-trivial normal subgroup of G.
   (b) Show that if a group G of order 28 has a normal subgroup of order 4, then G is abelian.

9. Let G be a finite group acting on a set X, and let $\mathrm{Fix}(g) = \{x \in X : g(x) = x\}$ be the set of points fixed by g. By counting the set $\{(g, x) \in G \times X : g(x) = x\}$ in two ways, show that the number of orbits of the action is equal to
$$\frac{1}{|G|} \sum_{g \in G} |\,\mathrm{Fix}(g)|.$$
   Deduce that if G acts transitively and $|X| > 1$, then there is some $g \in G$ with no fixed point.
   *How many distinct ways are there to colour the faces of a cube with three colours? Here, we consider two colourings to be distinct if one can not be obtained from the other via a rotation.

10. Let p be a prime and let G be a group of order p^2. By considering the conjugation action of G on itself, show that G is abelian. Furthermore, show that up to isomorphism there are just two groups of that order for each prime p.

*11. Let G be a (not necessarily finite) group generated by a finite set X. Prove that the number of subgroups of a given index n in G is finite, and give a bound for this number in terms of n and $|X|$.