IA Groups - Example Sheet 2

Michaelmas 2022 rdc26@cam.ac.uk

1. Write the following permutations as products of disjoint cycles, and compute their order and sign:
   (a) $(123)(1234)(132)$,
   (b) $(123)(235)(345)(45)$.

2. Show that S_n is generated by each of the following sets of permutations:
   (a) the set of transpositions $\{(j\ j+1) : 1 \leq j < n\}$,
   (b) the set of transpositions $\{(1\ k) : 1 < k \leq n\}$,
   (c) the set $\{(12), (123 \cdots n)\}$.

3. What is the largest possible order of an element in S_5? And in S_9? Show that every element in S_{10} of order 14 is odd.

4. Let H be a subgroup of S_n. Show that if H contains an odd element, then exactly half of its elements are odd.

5. (a) Show that A_4 has no subgroup of order 6.
   (b) Show that S_4 has a subgroup of order d for each divisor d of $|S_4|$. For which d does S_4 have two non-isomorphic subgroups of order d?

6. (a) Let G be a finite group and let K and H be subgroups of G, with $K \leqslant H$. Show that $|G : K| = |G : H| \cdot |H : K|$.
   (b) Let G be an infinite group, and let H and K be subgroups of finite index in G (i.e. $|G : K| < \infty$, $|G : H| < \infty$). Show that $H \cap K$ is also of finite index in G.

7. (a) Show that if a group G contains an element of order 6, and an element of order 10, then G has order at least 30.
   (b) Let G be a group of order 85, and let H be a subgroup of G containing at least 18 elements. Determine H.

8. Is it true that if K is a normal subgroup of H, and H is a normal subgroup of G, then K is a normal subgroup of G?

9. Let H be a subgroup of a group G. Find the largest normal subgroup of G contained in H in terms of H and the elements of G.

10. Show that $C_6 \cong C_2 \times C_3$. Explain why $C_{12} \not\cong C_6 \times C_2$. Express C_{12} as a direct product $C_n \times C_m$, with $n, m > 1$. When is $C_{nm} \cong C_n \times C_m$?

11. Show that the dihedral group D_{12} is isomorphic to the direct product $D_6 \times C_2$. Is D_{16} isomorphic to $D_8 \times C_2$?

12. Show that a group of order 10 is either cyclic or dihedral. Extend your proof to groups of order $2p$, with p an odd prime.

13. Must a group of order 55 have elements of order 5 and of order 11? Must a group of order 55 be cyclic? (Hint: Consider the functions $f : \mathbb{Z}_{11} \to \mathbb{Z}_{11}$ of the form $f(x) = ax + b$ where $a, b \in \mathbb{Z}_{11}$ and $a \neq 0$.) *Must a group of order 65 have elements of order 5 and of order 13? Must a group of order 65 be cyclic?