IA Groups - Example Sheet 1

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1. Let G be a group. Show that the identity e is the only element satisfying the equation $g^2 = g$ in G.

2. Let $G = \{x \in \mathbb{R} : x \neq -1\}$, and define $x * y := x + y + xy$ (where xy denotes the usual product of two real numbers). Show that $(G, *)$ is a group. What is the inverse 2^{-1} of the element 2 in G? Solve the equation $2 * x * 5 = 6$ in G.

3. Let H and K be subgroups of a group G. Prove that the intersection $H \cap K$ is a subgroup of G. Prove that the union $H \cup K$ is a subgroup of G if and only if either $H \subseteq K$ or $K \subseteq H$.

4. Let $X \subseteq G$. Show that the following definitions are equivalent.

   (i) $\langle X \rangle$ is the intersection of all subgroups containing X.

   (ii) $\langle X \rangle$ is the smallest subgroup containing X, i.e. $\langle X \rangle \leqslant H$ whenever $X \subseteq H \leqslant G$.

5. Let G be a finite group.

   (a) Let $g \in G$. Show that there is a positive integer n such that $g^n = e$. (The least such n is called the *order* of g.)

   (b) Show that there exists a positive integer n such that $g^n = e$ for all $g \in G$.

6. Let S be a finite non-empty set of non-zero complex numbers which is closed under multiplication. Show that S is a subset of the set $\{z \in \mathbb{C} : |z| = 1\}$. Show that S is a group with respect to multiplication, and deduce that for some $n \in \mathbb{N}$, S is the set of nth roots of unity, that is, $S = \{e^{2\pi ik/n} : k = 0, 1, \ldots, n-1\}$.

7. Let G be a group in which every element other than the identity has order two. Show that G is abelian. Show that if G is also finite, then the order of G is a power of 2. (Consider a minimal generating set for G.) Can such a group be infinite?

8. Let G be a group of even order. Show that G contains an element of order two. Can a group have exactly two elements of order two?
   * Which (not necessarily finite) groups have a non-zero even number of elements of order two?

9. Let G be a finite group and let $\theta : G \to H$ be a homomorphism to a group H. Let $g \in G$.

   (a) Show that the order of $\theta(g)$ is finite and divides the order of g.

   (b) Define the kernel of θ to be $\mathrm{Ker}(\theta) = \{g \in G : \theta(g) = e_H\}$. Prove that $\mathrm{Ker}(\theta)$ is a subgroup of G. Furthermore, suppose $g \in G$ and $k \in \mathrm{Ker}(\theta)$, show that $gkg^{-1} \in \mathrm{Ker}(\theta)$.

10. Let H and G be groups and let $X \subseteq G$ such that $\langle X \rangle = G$. Show that a homomorphism $G \to H$ is uniquely determined by its image on X: that is, if $\varphi : G \to H$ and $\psi : G \to H$ are homomorphisms such that $\varphi(x) = \psi(x)$ for all $x \in X$, then $\varphi(g) = \psi(g)$ for all $g \in G$.

11. Let C_n be the cyclic group of order n. If n is odd, show that the only homomorphism $\theta : D_{2n} \to C_n$ is the trivial homomorphism, i.e. $\theta(g) = e$ for all $g \in D_{2n}$. Find all homomorphisms $D_{2n} \to C_n$ when n is even. How many isomorphisms $C_n \to C_n$ are there?

12. Prove that every subgroup of a cyclic group is cyclic. Draw the subgroup lattice diagram for C_{24}.

*13. Is there an infinite group all of whose proper subgroups are finite?