

## IA Groups - Example Sheet 1

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1. Let  $G$  be a group. Show that the identity  $e$  is the only element satisfying the equation  $g^2 = g$  in  $G$ .
2. Let  $G = \{x \in \mathbb{R} : x \neq -1\}$ , and define  $x * y := x + y + xy$  (where  $xy$  denotes the usual product of two real numbers). Show that  $(G, *)$  is a group. What is the inverse  $2^{-1}$  of the element 2 in  $G$ ? Solve the equation  $2 * x * 5 = 6$  in  $G$ .
3. Let  $H$  and  $K$  be subgroups of a group  $G$ . Prove that the intersection  $H \cap K$  is a subgroup of  $G$ . Prove that the union  $H \cup K$  is a subgroup of  $G$  if and only if either  $H \subseteq K$  or  $K \subseteq H$ .
4. Sometimes a group is defined as follows. Let  $G$  be a set with a binary operation  $*$ . Suppose the associative law holds and there exists an element  $e \in G$  such that (i)  $g * e = g$  for all  $g \in G$  and (ii) for each  $g \in G$  there exists  $h \in G$  such that  $g * h = e$ . Prove that this is equivalent to the definition given in lectures.
5. Let  $G$  be a finite group.
  - (a) Let  $g \in G$ . Show that there is a positive integer  $n$  such that  $g^n = e$ . (The least such  $n$  is called the *order* of  $g$ .)
  - (b) Show that there exists a positive integer  $n$  such that  $g^n = e$  for all  $g \in G$ .
6. Let  $S$  be a finite non-empty set of non-zero complex numbers which is closed under multiplication. Show that  $S$  is a subset of the set  $\{z \in \mathbb{C} : |z| = 1\}$ . Show that  $S$  is a group with respect to multiplication, and deduce that for some  $n \in \mathbb{N}$ ,  $S$  is the set of  $n$ th roots of unity, that is,  $S = \{e^{2\pi i k/n} : k = 0, 1, \dots, n-1\}$ .
7. Let  $G$  be a group in which every element other than the identity has order two. Show that  $G$  is abelian. Show that if  $G$  is also finite, then the order of  $G$  is a power of 2. (Consider a minimal generating set for  $G$ .) Can such a group be infinite?
8. Let  $G$  be a group of even order. Show that  $G$  contains an element of order two. Can a group have exactly two elements of order two?  
\* Which (not necessarily finite) groups have a non-zero even number of elements of order two?
9. Let  $G$  be a finite group and let  $\theta : G \rightarrow H$  be a homomorphism to a group  $H$ . Let  $g \in G$ .
  - (a) Show that the order of  $\theta(g)$  is finite and divides the order of  $g$ .
  - (b) Define the kernel of  $\theta$  to be  $\text{Ker}(\theta) = \{g \in G : \theta(g) = e_H\}$ . Prove that  $\text{Ker}(\theta)$  is a subgroup of  $G$ . Furthermore, suppose  $g \in G$  and  $k \in \text{Ker}(\theta)$ , show that  $gk g^{-1} \in \text{Ker}(\theta)$ .
10. Let  $H$  and  $G$  be groups and let  $X \subseteq G$  such that  $\langle X \rangle = G$ . Show that a homomorphism  $G \rightarrow H$  is uniquely determined by its image on  $X$ : that is, if  $\varphi : G \rightarrow H$  and  $\psi : G \rightarrow H$  are homomorphisms such that  $\varphi(x) = \psi(x)$  for all  $x \in X$ , then  $\varphi(g) = \psi(g)$  for all  $g \in G$ .
11. Let  $C_n$  be the cyclic group of order  $n$ . If  $n$  is odd, show that the only homomorphism  $\theta : D_{2n} \rightarrow C_n$  is the trivial homomorphism, i.e.  $\theta(g) = e$  for all  $g \in D_{2n}$ . Find all homomorphisms  $D_{2n} \rightarrow C_n$  when  $n$  is even. How many isomorphisms  $C_n \rightarrow C_n$  are there?
12. Prove that every subgroup of a cyclic group is cyclic. Draw the subgroup lattice diagram for  $C_{24}$ .
- \*13. Is there an infinite group all of whose proper subgroups are finite?