IA Groups - Example Sheet 4

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1. Find the centre of S_n, and of A_n, for all n.

2. Given $\sigma \in A_n$, show that the conjugacy class of σ splits in A_n if and only if the cycles (including singletons) in the disjoint cycle decomposition of σ have distinct odd lengths.

3. Determine the sizes of the conjugacy classes in A_6. Deduce that A_6 is a simple group.

4. Construct a Möbius map that maps $\{z \in \mathbb{C} : |z-1| < 1\}$ onto $\{z \in \mathbb{C} : |z| > 2\} \cup \{\infty\}$.

5. Let G be the subgroup of Möbius maps that send the set $\{0, 1, \infty\}$ to itself. What are the elements of G? Which standard group is isomorphic to G? Find the group of Möbius maps that send the set $\{0, 2, \infty\}$ to itself, with as little calculation as possible.

6. Determine the conditions on $\lambda, \mu \in \mathbb{C}$ under which the Möbius maps $f(z) = \lambda z$ and $g(z) = \mu z$ are conjugate in $\mathcal{M}$.

7. Let G be the subset of $SL_3(\mathbb{R})$ consisting of all matrices of the form
$$\begin{pmatrix} a & 0 & 0 \\ b & w & x \\ c & y & z \end{pmatrix}.$$
Show that G is a subgroup of $SL_3(\mathbb{R})$. Construct a surjective homomorphism from G to $GL_2(\mathbb{R})$ and find its kernel.

8. Let G be the set of all matrices of the form
$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}.$$
Show that G is a subgroup of $GL_3(\mathbb{R})$. Let $H \subset G$ be the subset of those matrices with $a = c = 0$. Show that H is a normal subgroup of G and determine the quotient group G/H.

9. Find the orbits of the actions of the subgroups H and K of $GL_2(\mathbb{R})$ on $\mathbb{R}^2$, where
$H = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in GL_2(\mathbb{R}) \right\}$, and $K = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in GL_2(\mathbb{R}) \right\}$.

10. Show that the only normal subgroup of O_2 containing a reflection is O_2 itself.

11. (a) Find a surjective homomorphism from O_3 to C_2, and another from O_3 to SO_3.
    (b) Prove that $O_3 \cong SO_3 \times C_2$.
    (c) Is $O_4 \cong SO_4 \times C_2$?

12. Use cross-ratios to prove Ptolemy's Theorem: "For any quadrilateral whose vertices lie on a circle, the product of the lengths of the diagonals equals the sum of the products of the lengths of pairs of opposite sides."

13. Let $SL_2(\mathbb{R})$ act on $\hat{\mathbb{C}}$ via Möbius maps. Find the orbit and stabiliser of i and ∞. By considering the orbit of i under the action of the stabiliser of ∞, show that every $g \in SL_2(\mathbb{R})$ can be written as $g = hk$ with h an upper-triangular matrix (i.e. one with all coefficients below the diagonal equal to 0) and $k \in SO_2$.

14. Prove that S_n has a subgroup isomorphic to Q_8 if and only if $n \geq 8$.
    *Does $GL_2(\mathbb{R})$ have a subgroup isomorphic to Q_8?

*15. Let G be a finite non-trivial subgroup of SO_3. Let X be the set of points on the unit sphere in $\mathbb{R}^3$ which are fixed by at least one non-trivial rotation in G. Show that G acts on X and that the number of orbits is either 2 or 3. What is G if there are only two orbits?