IA Groups - Example Sheet 2

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1. Show that the dihedral group D_{12} is isomorphic to the direct product $D_6 \times C_2$. Is D_{16} isomorphic to $D_8 \times C_2$?

2. Write the following permutations as products of disjoint cycles, and compute their order and sign:
   (a) $(123)(1234)(132)$,
   (b) $(123)(235)(345)(45)$.

3. Show that S_n is generated by each of the following sets of permutations:
   (a) the set of transpositions $\{(j\ j+1) : 1 \leq j < n\}$,
   (b) the set of transpositions $\{(1\ k) : 1 < k \leq n\}$,
   (c) the set $\{(12), (123\cdots n)\}$.

4. What is the largest possible order of an element in S_5? And in S_9? Show that every element in S_{10} of order 14 is odd.

5. Let H be a subgroup of S_n. Show that if H contains an odd element, then exactly half of its elements are odd.

6. Show that S_4 has a subgroup of order d for each divisor d of $|S_4|$, and find two non-isomorphic subgroups of order 4. Give an example of a group G and a divisor d of $|G|$ such that G has no subgroup of order d.

7. (a) Let G be a finite group and let K and H be subgroups of G, with $K \leqslant H$. Show that $|G : K| = |G : H| \cdot |H : K|$.
   (b) Let G be a (possibly infinite) group, and let H and K be subgroups of finite index in G (i.e. $|G : K| < \infty$, $|G : H| < \infty$). Show that $H \cap K$ is also of finite index in G.

8. (a) Show that if a group G contains an element of order 6, and an element of order 10, then G has order at least 30.
   (b) Let G be a group of order 85, and let H be a subgroup of G containing at least 18 elements. Determine H.

9. What is the order of the Möbius map $f(z) = iz$? Which points are fixed by f, i.e. for which $z \in \hat{\mathbb{C}}$ do we have $f(z) = z$? If h is another Möbius map, find the order and fixed points of hfh^{-1}. Construct a Möbius map of order 4 that fixes 1 and -1.

10. Give an example to show that if K is a normal subgroup of H, and H is a normal subgroup of G, then K is not necessarily a normal subgroup of G.

11. Let H be a subgroup of a group G. Find the largest normal subgroup of G contained in H in terms of H and the elements of G.

12. Show that a group of order 10 is either cyclic or dihedral. *Extend your proof to groups of order $2p$, with p an odd prime.

*13. Must a group of order 55 have elements of order 5 and of order 11? Must a group of order 55 be cyclic? Must a group of order 65 have elements of order 5 and of order 13? Must a group of order 65 be cyclic?

*14. Which groups have exactly three subgroups?