

Part IA Groups // Example Sheet 4

1. Show that if $H \leq A_5$ then $|A_5/H| > 4$. [Hint: Consider an action on the set A_5/H .]
2. Let $G \subseteq SL_3(\mathbb{R})$ be the subset of all matrices of the form

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}.$$

Prove that G is a subgroup. Let $H \subset G$ be the subset of those matrices with $a = c = 0$. Show that H is a normal subgroup of G , and determine the quotient group G/H .

3. Let $G \subseteq SL_3(\mathbb{R})$ be the subset of all matrices of the form

$$\begin{bmatrix} a & 0 & 0 \\ b & c & d \\ e & f & g \end{bmatrix}.$$

Prove that G is a subgroup. Construct a surjective homomorphism $\phi : G \rightarrow GL_2(\mathbb{R})$, and find its kernel.

4. Show that matrices $A, B \in SL_2(\mathbb{C})$ are conjugate in $SL_2(\mathbb{C})$ if and only if they are conjugate in $GL_2(\mathbb{C})$. With a few exceptions—which you should find—show that matrices in $SL_2(\mathbb{C})$ are conjugate if and only if they have the same trace.
5. Let $SL_2(\mathbb{R})$ act on $\hat{\mathbb{C}}$ by Möbius transformations. Find the orbit and stabiliser of i and ∞ . By considering the orbit of i under the action of the stabiliser of ∞ , show that every $g \in SL_2(\mathbb{R})$ can be written as $g = hk$ with h upper triangular and $k \in SO(2)$. In how many ways can this be done?
6. Suppose that N is a normal subgroup of $O(2)$. Show that if N contains a reflection then $N = O(2)$.
7. Which pairs of elements of $SO(3)$ commute?
8. If $A \in M_{n \times n}(\mathbb{C})$ with entries A_{ij} , let $A^\dagger \in M_{n \times n}(\mathbb{C})$ have entries $\overline{A_{ji}}$. A matrix is called *unitary* if $AA^\dagger = I_n$. Show that the set $U(n)$ of unitary matrices is a subgroup of $GL_n(\mathbb{C})$. Show that

$$SU(n) = \{A \in U(n) \text{ s.t. } \det A = 1\}$$

is a normal subgroup of $U(n)$ and that $U(n)/SU(n) \cong S^1$. Show that Q_8 is isomorphic to a subgroup of $SU(2)$.

9. Let K be a normal subgroup of order 2 in a group G . Show that K is a subgroup of the centre $Z(G)$ of G . Show that if n is odd then $O(n) \cong SO(n) \times C_2$. Why doesn't a similar argument work if n is even?
10. Let $X = \{B \in M_{2 \times 2}(\mathbb{R}) \mid \text{Tr}(B) = 0\}$. Show that $A * B = ABA^{-1}$ defines an action of $SL_2(\mathbb{R})$ on X . Find the orbit and stabiliser of

$$B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Show that the set of matrices in X with determinant 0 is the union of three orbits.

11. * Prove that S_n has a subgroup isomorphic to Q_8 if and only if $n \geq 8$. Does $GL_2(\mathbb{R})$ have a subgroup isomorphic to Q_8 ?
12. * Let G be a finite non-trivial subgroup of $SO(3)$. Let

$$X = \{v \in \mathbb{R}^3 \text{ s.t. } |v| = 1 \text{ and there exists a } g \in G \setminus \{e\} \text{ with } g * v = v\}.$$

Show that G acts on X and that there are either 2 or 3 orbits. What is G if there are 2 orbits?