**Part IA Groups // Example Sheet 2**

1. Let $G$ be the group of all symmetries of a cube. Show that $G$ acts on the set of 4 lines joining diagonally opposite pairs of vertices. Show that if $\ell$ is one of these lines then $G_\ell \cong D_6 \times C_2$.

2. Let $H$ be a subgroup of a group $G$. Show that there is a bijection between the set of left cosets of $H$ in $G$ and the set of right cosets of $H$ in $G$.

3. If $G$ is a finite group, $H$ is a subgroup of $G$, and $K$ is a subgroup of $H$, show that $|G/K| = |G/H| \cdot |H/K|$.

4. Show that if a group $G$ contains an element of order 6, and an element of order 10, then $G$ has order at least 30.

5. Show that $D_{2n}$ has one conjugacy class of reflections if $n$ is odd and two conjugacy classes of reflections if $n$ is even.

6. Let $G$ be a finite group and let $\mathrm{Sub}(G)$ be the set of all its subgroups. Show that $g * H := gHg^{-1}$ defines an action of $G$ on $\mathrm{Sub}(G)$. Show that for $H \in \mathrm{Sub}(G)$ the size of the orbit of $H$ under this action is at most $|G/H|$. Deduce that if $H \neq G$ then $G$ is not the union of all conjugates of $H$.

7. Suppose that $G$ acts on $X$ and that $y = g \cdot x$ for some $x, y \in X$ and $g \in G$. Show that $G_y = gG_xg^{-1}$.

8. Let $G$ be a finite abelian group acting faithfully on a set $X$. Show that if the action is transitive then $|G| = |X|$.

9. Consider the Möbius transformations $f(z) = e^{2\pi i/n}z$ and $g(z) = 1/z$. Show that the subgroup $G$ of the Möbius group $\mathcal{M}$ generated by $f$ and $g$ is isomorphic to $D_{2n}$.

10. Express the Möbius transformation $f(z) = \frac{2z+3}{z-4}$ as the composition of tranformations of the form $z \mapsto az$, $z \mapsto z + b$ and $z \mapsto 1/z$. Hence show that $f$ sends the circle described by $|z - 2i| = 2$ onto the circle described by $|8z + (6 + 11i)| = 11$.

11. Let $G$ be the subgroup of Möbius transformations that send the set $\{0, 1, \infty\}$ to itself. What are the elements of $G$? Which standard group is isomorphic to $G$? What is the group of Möbius transformations that send the set $\{0, 2, \infty\}$ to itself.

12. Prove or disprove each of the following statements:

    (i) The Möbius group is generated by Möbius transformations of the form $z \mapsto az$ and $z \mapsto z + b$.

    (ii) The Möbius group is generated by Möbius transformations of the form $z \mapsto az$ and $z \mapsto 1/z$.

    (iii) The Möbius group is generated by Möbius transformations of the form $z \mapsto z+b$ and $z \mapsto 1/z$.

13. Show that any invertible function $f\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ that preserves the cross-ratio, i.e. such that

    $$[z_1, z_2, z_3, z_4] = [f(z_1), f(z_2), f(z_3), f(z_4)] \text{ for all distinct } z_1, z_2, z_3, z_4 \in \hat{\mathbb{C}},$$

    is a Möbius transformation.

14. Determine under what conditions on $\lambda, \mu \in \mathbb{C}$ the Möbius transformations $f(z) = \lambda z$ and $g(z) = \mu z$ are conjugate in $\mathcal{M}$.

15. What is the order of the Möbius transformation $f(z) = iz$? What are its fixed points? If $h$ is another Möbius transformation what can you say about the order and the fixed points of $hfh^{-1}$? Construct a Möbius transformation of order 4 that fixes 1 and $-1$.

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