Groups Example Sheet 4

Michaelmas 2016 Julia Goedecke

Please send comments and corrections to jg352.

1. What is the order of the Möbius map $f(z) = iz$? If h is any Möbius map, find the order of hfh^{-1} and its fixed points. Use this to construct a Möbius map of order four that fixes 1 and -1.

2. Suppose that the group G acts on the set X. Let $x \in X$, let $y = g(x)$ for some $g \in G$. Show that the stabiliser G_y equals the conjugate gG_xg^{-1} of the stabiliser G_x.

3. Show that if H is a proper subgroup of index n in A_5 then $n > 4$. [Consider the left coset action of A_5 on the set of left cosets of H in A_5.]

4. What is the group of all rotational symmetries of a Toblerone box, a solid triangular prism with an equilateral triangle as a cross-section, with ends orthogonal to the longitudinal axis of the prism? And the group of all symmetries?

5. Let $g(z) = (z+1)/(z-1)$. By considering the points $g(0)$, $g(\infty)$, $g(1)$ and $g(i)$, find the image of the real axis $\mathbb{R}$ and of the imaginary axis $\mathbb{I}$ under g. What is $g(\Sigma)$, where Σ is the first quadrant in $\mathbb{C}$?

6. (a) Construct a Möbius map that maps $\{z \in \mathbb{C} \mid |z-1| < 1\}$ onto $\{z \in \mathbb{C} \mid |z| > 2\}$.

   (b) Construct a Möbius map that maps the strip $\{z \in \mathbb{C} \mid 0 < \mathrm{Im}(z) < 1\}$ onto the region between the circles $|z-1| = 1$ and $|z-2| = 2$.

7. Let G be the group of Möbius transformations which map the set $\{0, 1, \infty\}$ onto itself. Find all the elements in G. To which standard group is G isomorphic? Justify your answer.

   Find the group of Möbius transformations which map the set $\{0, 2, \infty\}$ onto itself. [Try to do as little calculation as possible.]

8. Let Q_8 be the quaternion group. Find all of the subgroups of Q_8, and show that they are all normal. Find the conjugacy classes, and show that the centre $Z = Z(Q_8)$ has order 2. To what standard group is the quotient group Q_8/Z isomorphic? Compare to the quotient $D_8/Z(D_8)$ of the dihedral group of order 8 quotiented by its centre. What do you conclude?

9. Let N be a normal subgroup of the orthogonal group O_2). Show that if N contains a reflection in some line through the origin, then $N = \mathrm{O}_2$.

10. **Ptolemy's Theorem** states: "For any quadrilateral whose vertices lie on a circle, the product of the lengths of the diagonals equals the sum of the products of the lengths of pairs of opposite sides."

    Prove Ptolemy's Theorem using complex numbers and cross-ratios.

11. Let G be the general linear group $\mathrm{GL}_2(5)$ of invertible 2×2 matrices over the field $\mathbb{F}_5$ of integers modulo 5, so that the arithmetic in G is modulo 5. Consider the subgroup $\mathrm{SL}_2(5)$ of G consisting of matrices of determinant 1. Show that G has order 480. By considering a suitable homomorphism from G to another group, deduce that $\mathrm{SL}_2(5)$ has order 120.

    Prove that $-I$ is the only element of $\mathrm{SL}_2(5)$ of order 2, and deduce that $\mathrm{SL}_2(5)$ has no subgroup isomorphic to A_5.

The starred and exploration questions are not necessarily harder, but not necessary for a good understanding of the course. They should only be attempted once you have a solid understanding of the core material. They should also not be attempted to the detriment of later example sheets, or other courses. Exploration questions are meant to lead you as far as you are interested: just start and see how far you can get. There is not necessarily a "full solution".

12. * (Follow-on from Question 11) Find a subgroup of $\mathrm{SL}_2(5)$ isomorphic to Q_8, and an element of order 3 normalising it in $\mathrm{SL}_2(5)$. Deduce that $\mathrm{SL}_2(5)$ has a subgroup of index 5, and obtain a homomorphism from $\mathrm{SL}_2(5)$ to S_5.

    Deduce that $\mathrm{SL}_2(5)/\{\pm I\}$ is isomorphic to the alternating group A_5.

13. * Let Q_8 be the quaternion group again. For a group G, write Z for its centre.
    (a) Prove that no group G satisfies $G/Z \cong Q_8$.
    (b) Prove that S_n has a subgroup isomorphic to Q_8 iff $n \geq 8$.
    (c) Does $\mathrm{GL}_2(\mathbb{R})$ have a subgroup isomorphic to Q_8?

14. * Let G be a finite non-trivial subgroup of SO_3. Let X be the set of points on the *unit sphere* in $\mathbb{R}^3$ fixed by at least one non-trivial rotation in G. Show that G acts on X and that the number of orbits is either 2 or 3. What is G if there are only two orbits?