IA Groups: Example Sheet 2

1. Show that if a group G contains an element of order six, and an element of order ten, then G has order at least 30.
2. Show that the set $\{1, 3, 5, 7\}$ with multiplication modulo 8 is a group. Is this group isomorphic to C_4 or $C_2 \times C_2$? Justify your answer.
3. Let G be a finite group and θ a homomorphism from G to H. Let $g \in G$. Show that the order of $\theta(g)$ is finite and divides the order of g.
4. Let H be a subgroup of the group G. Find a (natural) bijection between the set of all left cosets and the set of all right cosets of H in G.
5. Let H be a subgroup of the (finite) group G, let K be a subgroup of H. Show that the index $|G : K|$ equals the product $|G : H||H : K|$.
6. Let G be a subgroup of the symmetric group S_n. Show that if G contains any odd permutations then precisely half of the elements of G are odd.
7. Show that any subgroup of a cyclic group is cyclic. Find all the subgroups of the cyclic group C_n.
8. Show that the symmetric group S_4 has a subgroup of order d for each divisor d of 24, and find two non-isomorphic subgroups of order 4.

   Show that the alternating group A_4 has a subgroup of each order up to 4, but there is no subgroup of order 6.
9. List all the subgroups of the dihedral group D_8, and indicate which pairs of subgroups are isomorphic.

   Repeat for the quaternion group Q_8.
10. Let G be a group. If H is a normal subgroup of G and K is a normal subgroup of H, is K a normal subgroup of G?
11. Let K be a normal subgroup of index m in the group G. Show that $g^m \in K$ for any element $g \in G$.
12. Show that any group of order 10 is either cyclic or dihedral.
13. Show that the dihedral group D_{12} is isomorphic to the direct product $D_6 \times C_2$.
14. A finite group G is generated by a set T of elements of G if each element of G can be written as a finite product (possibly with repetitions) of powers of elements of T. Show that the symmetric group S_n is generated by each of the following sets of permutations:

    (i) the set $\{(j, k) : 1 \leq j < k \leq n\}$ of all transpositions in S_n;

    (ii) the set $\{(j, j + 1) : 1 \leq j < n\}$;

    (iii) the set $\{(1, k) : 1 < k \leq n\}$;

    (iv) the set $\{(1, 2), (12 \ldots n)\}$ consisting of a transposition and an n-cycle.

15. Consider a pack of $2n$ cards, numbered from 0 to $2n-1$. An *outer perfect shuffle* is a shuffle of the cards, in which one first splits the pack in two halves of equal sizes and then interleaves the cards of the two halves in such a way that the top and bottom card remain in the top and bottom position. Show that the order of the outer shuffle is the multiplicative order of 2 modulo $2n-1$.

    Deduce that after at most $2n-2$ repetitions of the outer shuffle we get the cards in the pack into the original position.

    What is the actual order of the outer shuffle of the usual pack of 52 cards?

    (There is also an *inner perfect shuffle* which differs from the outer shuffle in that the interleaving of the cards of the two halves is done so that neither the top nor the bottom card remains in the same position. What is the order of this shuffle of the usual pack of 52 cards?)

Comments and corrections should be sent to `rdc26@dpmms.cam.ac.uk`.