Michaelmas Term 2012 R. Camina

IA Groups: Example Sheet 1

1. Let G be any group. Show that the identity e is the unique solution of the equation $x^2 = x$.

2. Let H_1 and H_2 be two subgroups of the group G.
   Show that the intersection $H_1 \cap H_2$ is a subgroup of G.
   Show that the union $H_1 \cup H_2$ is a subgroup of G if and only if one of the H_i contains the other.

3. Show that the set of functions on $\mathbb{R}$ of the form $f(x) = ax + b$, where a and b are real numbers and $a \neq 0$, forms a group under composition of functions. Is this group abelian?

4. Let G be a finite group.
   (i) Let $g \in G$. Show that there is a positive integer n such that g^n equals e, the identity element. (The least such positive n is the *order* of g.)
   (ii) Show that there exists a positive integer n such that $g^n = e$ for all $g \in G$. (The least such positive n is the *exponent* of G.)

5. Show that the set G of complex numbers of the form $\exp(i\pi t)$ with t rational is a group under multiplication (with identity 1). Show that G is infinite, but that every element g of G has finite order.

6. Let S be a finite non-empty set of non-zero complex numbers which is closed under multiplication. Show that S is a subset of the set $\{z \in \mathbb{C} : |z| = 1\}$. Show that S is a group, and deduce that for some $n \in \mathbb{N}$, S is the set of n-th roots of unity; that is, $S = \{\exp(2k\pi i/n) : k = 0, \ldots, n-1\}$.

7. Let $G = \{x \in \mathbb{R} : x \neq -1\}$, and let $x * y = x + y + xy$, where xy denotes the usual product of two real numbers. Show that $(G, *)$ is a group. What is the inverse 2^{-1} of 2 in this group? Solve the equation $2 * x * 5 = 6$.

8. Let G and H be groups and $\theta : G \to H$ a group homomorphism. Define the kernel of θ to be $\mathrm{Ker}(\theta) = \{g \in G : \theta(g) = e_H\}$. Prove that $\mathrm{Ker}(\theta)$ is a subgroup of G. Furthermore, suppose $g \in G$ and $k \in \mathrm{Ker}(\theta)$, show that $gkg^{-1} \in \mathrm{Ker}(\theta)$.

9. Write the following permutations as products of disjoint cycles and compute their order and sign:
   (a) $(12)(1234)(12)$;
   (b) $(123)(45)(16789)(15)$.

10. What is the largest possible order of an element in S_5? And in S_9?
    Show that every element in S_{10} of order 14 is odd.

11. Let G be a group in which every element other than the identity has order two. Show that G is abelian. Show also that if G is finite, the order of G is a power of 2. (Consider a minimal generating set.)

12. Let G be a group of even order. Show that G contains an element of order two.

13. A *fifteen puzzle* consists of fifteen small square tiles, numbered 1 to 15, which are mounted in a 4×4 frame in such a way that each tile can slide vertically or horizontally into an adjacent square (if it is not already occupied by another tile), but the tiles cannot be lifted out of the tray. On the packet in which the puzzle was sold, it is asserted that it is impossible to manoeuvre the tiles from the first to the second of the configurations shown below. The packet is too small to contain a proof. Is the assertion true? Prove it.

1	2	3	4		15	14	13	12
5	6	7	8		11	10	9	8
9	10	11	12		7	6	5	4
13	14	15	-		3	2	1	-

Comments and corrections should be sent to `rdc26@dpmms.cam.ac.uk`.