IA Groups: Example Sheet 4

1. Consider the Möbius maps $f(z) = e^{2\pi i/n}z$ and $g(z) = 1/z$. Show that the subgroup G of the Möbius group $\mathcal{M}$ generated by f and g is a dihedral group of order $2n$.

2. Let $g(z) = (z+1)/(z-1)$. By considering the points $g(0)$, $g(\infty)$, $g(1)$ and $g(i)$, find the image of the real axis $\mathbb{R}$ and of the imaginary axis $\mathbb{I}$ under g. What is $g(\Sigma)$, where Σ is the first quadrant in $\mathbb{C}$?

3. What is the order of the Möbius map $f(z) = iz$? If h is any Möbius map, find the order of hfh^{-1} and its fixed points. Use this to construct a Möbius map of order four that fixes 1 and -1.

4. Let G be the set of all 3×3 matrices of the form

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix},$$

with $x, y, z \in \mathbb{R}$. Show that G is a subgroup of the group of invertible real matrices under multiplication. Let H be the subset of G given by those matrices with $x = z = 0$. Show that H is a normal subgroup of G and identify G/H.

5. Show that the set $SL_2(\mathbb{Z})$ of all 2×2 matrices of determinant 1 with integer entries is a group under multiplication.

6. Let G be the group of Möbius transformations which map the set $\{0, 1, \infty\}$ onto itself. Find all the elements in G. To which standard group is G isomorphic? Justify your answer.

   Find the group of Möbius transformations which map the set $\{0, 2, \infty\}$ onto itself. [Try to do as little calculation as possible.]

7. Let G be as in the previous question. Show that, given $\sigma \in S_4$, there exists $f_\sigma \in G$ for which, whenever z_1, z_2, z_3 and z_4 are four distinct points in $\mathbb{C}_\infty$, we have $f_\sigma([z_1, z_2, z_3, z_4]) = [z_{\sigma(1)}, z_{\sigma(2)}, z_{\sigma(3)}, z_{\sigma(4)}]$.

   Show that the map $\sigma \mapsto f_{\sigma^{-1}}$ from S_4 to G gives a homomorphism from S_4 onto S_3. Find its kernel.

8. The *centre* of a group G consists of all those elements of G that commute with all the elements of G. Show that the centre Z of the general linear group $GL_2(\mathbb{C})$ consists of all scalar matrices. Identify the centre of the special linear group $SL_2(\mathbb{C})$.

9. Let G be the set of all 3×3 real matrices of determinant 1 of the form

$$\begin{pmatrix} a & 0 & 0 \\ b & x & y \\ c & z & w \end{pmatrix}.$$

Verify that G is a group. Find a homomorphism from G onto the group $GL_2(\mathbb{R})$ of all non-singular 2×2 real matrices, and find its kernel.

10. When do two elements of SO_3 commute?

11. Let K be a normal subgroup of order 2 in the group G. Show that K lies in the centre of G, that is $kg = gk$ for all $k \in K$ and $g \in G$.

Describe a surjective homomorphism of the orthogonal group $O(3)$ onto C_2 and another onto the special orthogonal group $SO(3)$.

12. If A is a complex $n \times n$ matrix with entries a_{ij}, let A^* be the complex $n \times n$ matrix $\bar{A}^t$ with entries $\overline{a_{ji}}$. The matrix A is called *unitary* if $AA^* = I$. Show that the set $U(n)$ of unitary matrices forms a group under matrix multiplication. Show that

$$SU(n) = \{A \in U(n) : \det A = 1\}$$

is a normal subgroup of $U(n)$ and that $U(n)/SU(n)$ is isomorphic to S^1, the group of the unit circle in $\mathbb{C}$ under multiplication.

Show that $SU(2)$ contains the quaternion group Q_8 as a subgroup.

13. Let G be the special linear group $SL_2(5)$ of 2×2 matrices of determinant 1 over the field $\mathbb{F}_5$ of integers modulo 5, so that the arithmetic in G is modulo 5. Show that G is a group of order 120. Prove that $-I$ is the only element of G of order 2.

Find a subgroup of G of order 8 isomorphic to Q_8, and an element of order 3 normalising it in G. Deduce that G has a subgroup of index 5, and obtain a homomorphism from G to S_5. Deduce that $SL_2(5)/\{\pm I\}$ is isomorphic to the alternating group A_5. Show that $SL_2(5)$ has no subgroup isomorphic to A_5.

Comments and corrections should be sent to `saxl@dpmms.cam.ac.uk`.