# GROUPS EXAMPLES 3

**G.P. Paternain Michaelmas 2007**

The questions on this sheet are not all equally difficult and the harder ones are marked with $*$'s. Comments on and/or corrections to the questions on this sheet are always welcome, and may be e-mailed to me at `g.p.paternain@dpmms.cam.ac.uk`.

**1**. If $H$ is a subgroup of a finite group $G$ and $G$ has twice as many elements as $H$, show that $H$ is normal in $G$.

**2**. Let $H$ be a subgroup of the cyclic group $C_n$. What is $C_n/H$?

**3**. Show that every subgroup of rotations in the dihedral group $D_{2n}$ is normal.

**4**. Show that a subgroup $H$ of a group $G$ is normal if and only if it is a union of conjugacy classes.

**5**. We know that in an abelian group every subgroup is normal. Now, let $G$ be a group in which every subgroup is normal, is it true that $G$ must be abelian?

**6**. Show that $\mathbb{Q}/\mathbb{Z}$ is an infinite group in which every element has finite order.

**7**. Let $G$ be the set of all $3 \times 3$ matrices of the form

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix},$$

with $x, y, z \in \mathbb{R}$. Show that $G$ is a subgroup of the group of invertible real matrices under multiplication. Let $H$ be the subset of $G$ given by those matrices with $x = z = 0$. Show that $H$ is a normal subgroup of $G$ and find $G/H$. [Use the isomorphism theorem.]

**8**. Consider the additive group $\mathbb{C}$ and the subgroup $\Gamma$ consisting of all *Gaussian integers* $m+in$, where $m, n \in \mathbb{Z}$. By considering the map

$$x + iy \mapsto (e^{2\pi i x}, e^{2\pi i y}),$$

show that the quotient group $\mathbb{C}/\Gamma$ is isomorphic to the torus $S^1 \times S^1$.

**9**. Let $G$ be a finite group and $H \neq G$ a subgroup. Let $k$ be the cardinality of the set of left cosets of $H$ ($k$ is sometimes called the *index* of $H$) and suppose that $|G|$ does not divide $k!$. Show that $H$ contains a non-trivial normal subgroup of $G$. [Let $G$ act on the set of left cosets.] Show that a group of order 28 has a normal subgroup of order 7.

**10**. Show that if a group $G$ of order 28 has a normal subgroup of order 4, then $G$ is abelian.

**11**. Let $H$ be a subgroup of a group $G$. Show that $H$ is a normal subgroup of $G$ if and only if there is some group $K$, and some homomorphism $\theta : G \to K$, whose kernel is $H$.

**12**. Let $GL(2,\mathbb{R})$ be the group of all $2 \times 2$ invertible matrices and let $SL(2,\mathbb{R})$ be the subset of $GL(2,\mathbb{R})$ consisting of matrices of determinant 1. Show that $SL(2,\mathbb{R})$ is a normal subgroup of $GL(2,\mathbb{R})$. Show that the quotient group $GL(2,\mathbb{R})/SL(2,\mathbb{R})$ is isomorphic to the multiplicative group of non-zero real numbers.

**13**. Let $G$ be a subgroup of the group of isometries of the plane. Show that the set $T$ of translations in $G$ is a normal subgroup of $G$ ($T$ is called the *translation subgroup*). [If we think of the plane as $\mathbb{C}$ you may assume that all isometries have the form $z \mapsto az + b$ or $z \mapsto a\bar{z} + b$, where $a$ and $b$ are complex numbers and in both cases $|a| = 1$.]

**14**$^*$. A *frieze group* is a group $F$ of isometries of $\mathbb{C}$ that leaves the real line invariant and whose translation subgroup $T$ is infinite cyclic. If $F$ is a frieze group, classify $F/T$.