GROUPS EXAMPLES 2

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The questions on this sheet are not all equally difficult and the harder ones are marked with $*$'s. Comments on and/or corrections to the questions on this sheet are always welcome, and may be e-mailed to me at `g.p.paternain@dpmms.cam.ac.uk`.

1. Show that if a group G contains an element of order six, and an element of order ten, then G has order at least 30.

2. Show that the set $\{1, 3, 5, 7\}$, with multiplication modulo 8 is a group. Is this group isomorphic to C_4 or $C_2 \times C_2$?

3. How many subgroups of order four does the quaternion group $\mathbb{H}_8$ have?

4. Let H be a subgroup of a group G. Show that if $aH = bH$ then $Ha^{-1} = Hb^{-1}$. Use this to show that there is a bijection between the set of left cosets and the set of right cosets of H.

5. What is the order of the Möbius map $f(z) = iz$? If h is any Möbius map, find the order of hfh^{-1} and its fixed points. Use this to construct a Möbius map of order four that fixes 1 and -1.

6. Show that $\rho(t, (x, y)) = (e^t x, e^{-t} y)$, where $t \in \mathbb{R}$ and $(x, y) \in \mathbb{R}^2$, defines an action of $(\mathbb{R}, +)$ on $\mathbb{R}^2$. Describe the orbits of the action and find all possible stabilizers. There is a differential equation in the plane that is satisfied by all orbits, can you find it?

7. Suppose that G acts on X and that $y = gx$, where $x, y \in X$ and $g \in G$. Show that $\mathrm{Stab}(y) = g\,\mathrm{Stab}(x)\,g^{-1}$ (hence, the stabilizers of two points in the same orbit are conjugate).

8. Let G be a finite group and let X be the set of all it subgroups. Given $g \in G$ and $H \in X$ show that $(g, H) \mapsto gHg^{-1}$ defines an action of G on X. Show that the orbit of H has at most $|G|/|H|$ elements. If $H \neq G$, show that there is an element in G which does not belong to any conjugate of H.

9. Let G be a finite group acting faithfully on a finite set X. Show that if G is abelian and there is only one orbit, then $|G| = |X|$.

10. Show that D_{2n} (the group of symmetries of the regular n-gon) has one conjugacy class of reflections if n is odd, and two conjugacy classes of reflections if n is even.

11. Let G be the group of all symmetries of the cube and let ℓ be the line between two diagonally opposite vertices. Let $H = \{g \in G : \; g\ell = \ell\}$. Show that H is a subgroup of G isomorphic to $S_3 \times C_2$.

12. Classify all groups of order 10.

13. Let S^1 denote the unit circle in $\mathbb{C}$ and let $S^3 = \{(w_1, w_2) \in \mathbb{C}^2 : \; |w_1|^2 + |w_2|^2 = 1\}$. Show that given $(t_1, t_2) \in S^1 \times S^1$ and $(w_1, w_2) \in S^3$,

$$((t_1, t_2), (w_1, w_2)) \mapsto (t_1 w_1, t_2 w_2)$$

defines an action of the torus $S^1 \times S^1$ on S^3. Describe the orbits and find all stabilizers.

14*. Let p be a prime. Show that every group of order p^2 is abelian. [Hint: consider the action of the group on itself by conjugation.]