Analysis I **Lent term 2020**

Example Sheet 1

1. Let (x_n) be a real sequence with $x_n \neq 0$ for all n. If $x_n \to \infty$, show that $\frac{1}{x_n} \to 0$. If $x_n \to 0$, does it follow that $\frac{1}{x_n} \to \infty$? Now suppose that $x_n \to \infty$ and that (y_n) is a real sequence with $y_n \to y \in \mathbb{R}$. If $y > 0$, prove that $x_n y_n \to \infty$.

2. Prove that $x^n \to 0$ for $x \in (0,1)$. (Do not assume the existence of the logarithm.)

3. Sketch the graphs of $y = x$ and $y = (x^4+1)/3$ and thereby illustrate the behavior of the real sequence (x_n) where $x_{n+1} = (x_n^4 + 1)/3$. For which of the three starting cases $x_1 = 0$, $x_1 = 1$ and $x_1 = 2$ does the sequence converge? Prove your assertion.

4. Let $x_1 > y_1 > 0$ and define $x_{n+1} = (x_n + y_n)/2$, $y_{n+1} = 2x_n y_n/(x_n + y_n)$ for $n \geq 1$. Show that $x_n > x_{n+1} > y_{n+1} > y_n$ and deduce that the two sequences converge to a common limit. What is the limit?

5. The real sequence (x_n) is bounded but does not converge. Prove that it has two convergent subsequences with different limits.

6. For $\alpha \in \mathbb{R}$, let $\{\alpha\} = \alpha - \lfloor \alpha \rfloor$ be the fractional part of α. Suppose $\alpha \in \mathbb{R}$ is an irrational number, and define $x_n = \{n\alpha\}$. For any $x \in [0,1]$, show that (x_n) has a subsequence converging to x.

7. For a fixed value of $t > 0$, define $\exp(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!}$, and let $x_n = (1 + \frac{t}{n})^n$. Using the binomial theorem, show that the sequence (x_n) is monotone increasing and is bounded above by $\exp(t)$. Show further that $x_n \to \exp(t)$.

8. Investigate the convergence of the following series. For those expressions containing the complex number z, find the values of z for which the series converges.

$$\sum_n \frac{\sin n}{n^2} \qquad \sum_n \frac{n^2 z^n}{5^n} \qquad \sum_n \frac{(-1)^n}{4 + \sqrt{n}} \qquad \sum_n \frac{z^n(1-z)}{n}$$

9. Show that the series $\sum_{n\geq 2} \frac{1}{n(\log n)^\alpha}$ converges for $\alpha > 1$ and diverges otherwise. Does $\sum_{n \geq 3} \frac{1}{n \log n \log\log n}$ converge?

10. The two series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots$ and $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \ldots$ have the same terms in different orders. Let s_n and t_n be the nth partial sums of these two series. Set $h_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots \frac{1}{n}$. Show that $s_{2n} = h_{2n} - h_n$ and $t_{3n} = h_{4n} - \frac{1}{2}h_{2n} - \frac{1}{2}h_n$. Show that (s_n) tends to a limit $s > 0$ and (t_n) tends to $\frac{3}{2}s$.

11. Let $z \in \mathbb{C}$. Show that the series

$$\frac{z}{1-z^2} + \frac{z^2}{1-z^4} + \frac{z^4}{1-z^8} + \frac{z^8}{1-z^{16}} + \dots$$

converges to $z/(1-z)$ if $|z| < 1$, converges to $1/(1-z)$ if $|z| > 1$, and diverges if $|z| = 1$.

12. Prove that every real sequence has a monotone subsequence. Deduce the Bolzano-Weierstrass theorem.

13. Let x be a real number and suppose the real series $\sum_n a_n$ converges, but does not converge absolutely. Prove the terms can be reordered so the resulting series converges to x. That is, there is a bijection $\pi : \mathbb{N} \to \mathbb{N}$ such that $\sum_n a_{\pi(n)} = x$.

14. Can the open interval $(0, 1)$ be written as a disjoint union of closed intervals of positive length?

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