ANALYSIS I—EXAMPLES 4

(updated 15 March 2026)

Exercises

1a. Let $f, g\colon I = [a,b] \to \mathbb{R}$. Show that (a) $\sup_I(f+g) \le \sup_I f + \sup_I g$ and $\inf_I(f+g) \ge \inf_I f + \inf_I g$; (b) $\sup_I(-f) = -\inf_I f$; (c) $\sup_I(\lambda f) = \lambda \sup_I f$ and $\inf_I(\lambda f) = \lambda \inf_I f$ if $\lambda > 0$ is fixed. Deduce that, if f and g are integrable, then $f + \mu g$ is integrable for any fixed $\mu \in \mathbb{R}$, and write a formula relating the different integrals.

1b. Let $f\colon \mathbb{R} \to \mathbb{R}$, and suppose f is integrable on $[-R, R]$ for each $R < \infty$. We say that the Cauchy principal value of f exists if $\text{p.v.}(f) = \lim_{R\to\infty} \int_{-R}^{R} f(x)dx$ exists (and is finite). Show that, if the improper integral $\int_{-\infty}^{\infty} f$ exists, then p.v.(f) exists. Is the converse true?

1c. Consider the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$. Show that if $|c_{n+1}/c_n| \to \infty$ or $|c_n|^{1/n} \to \infty$ as $n \to \infty$, then the power series only converges at a.

1d. Using the familiar properties of e^x from your pre-university education, show that e^x has Taylor series $\sum_{n=0}^{\infty} x^n/n!$ at $x = 0$. Repeat this exercise for one of the following functions: $\log(1+x)$, $\cos x$, or $\sin x$.

Problems

2. Let $f : [0,1] \to \mathbb{R}$ be continuous. Let

$$G(x,t) = \begin{cases} t(x-1), & t \le x \\ x(t-1), & t \ge x \end{cases}$$

Let $g(x) = \int_0^1 f(t)G(x,t)dt$. Show that $g''(x)$ exists for $x \in (0,1)$ and equals $f(x)$.

3. This question takes you through the construction of bump functions and partitions of unity.

(a) Starting from the function $h\colon \mathbb{R} \to \mathbb{R}$ defined by $h(x) = \exp(-1/x^2)$ for $x \neq 0$ and $h(0) = 0$ studied before in sheet 3, construct a function $g : \mathbb{R} \to \mathbb{R}$ that is infinitely-differentiable, positive on a given interval (a,b) and zero elsewhere.

(b) ($\star$) Assuming standard results concerning integration, including the fundamental theorem of calculus, construct a function from $\mathbb{R}$ to $\mathbb{R}$ that is infinitely-differentiable, identically 1 on $[-1,1]$ and identically 0 outside $(-2,2)$.

(c) ($\star$) Construct an infinitely-differentiable, non-negative, function $\psi : \mathbb{R} \to \mathbb{R}$ that is identically 0 outside $(-2,2)$, and satisfies

$$\sum_{n=-\infty}^{\infty} \psi(x-n) = 1, \qquad \text{for all } x \in \mathbb{R}.$$

4. Let $f : [a,b] \to \mathbb{R}$ be continuous and suppose that $\int_a^b f(x)g(x)\,dx = 0$ for every continuous function $g : [a,b] \to \mathbb{R}$ which vanishes near a and b. Must f vanish identically? [We say g vanishes near a and b if there exists $\epsilon > 0$ such that $g(x) = 0$ for $x \notin (a+\epsilon, b-\epsilon)$.]

5. Do these improper integrals converge?

$$\text{(i)} \int_1^{\infty} \sin^2(1/x)dx, \qquad \text{(ii)} \int_0^{\infty} x^p \exp(-x^q)dx,$$

where $p, q > 0$. [You may assume familiar results about the continuity and differentiability properties of the exponential and logarithm.]

6. This question concerns tests for determining convergence of improper integrals over an unbounded domain. Let $f\colon [a,\infty) \to \mathbb{R}$ be integrable on $[a,R]$ for every $R \geq a$.

(a) (Dirichlet test) Suppose f is continuous. Let $F(x) = \int_a^x f(t)dt$ for $x \in [a,\infty)$ and assume F is bounded ($\sup_{x\geq a} |F(x)| < \infty$). Let $g\colon [a,\infty) \to \mathbb{R}$ be monotone, continuously differentiable, and satisfy $g(x) \to 0$ as $x \to \infty$. Show that the improper integral $\int_a^\infty fg$ converges. Deduce that $\int_1^\infty x^{-1/2} \sin x \, dx$ converges.

(b) (Root test) Suppose $L = \lim_{x\to\infty} |f(x)|^{1/x}$ exists. What can you say about the convergence of the improper integral $\int_a^\infty |f|$ for $L > 1$, $L < 1$ and $L = 1$? ($\star$) What about $\int_a^\infty f$? Justify your answers. [You may assume the familiar properties of the exponential function.]

(c) ($\star$) Does the nth term test hold for improper integrals: if $f \geq 0$ and the improper integral $\int_a^\infty f$ converges, must $f(x) \to 0$ as $x \to \infty$? What if f is continuous?

7. Find the radius of convergence of each of the following power series.

$$\text{(i)} \sum_{n\geqslant 0} \frac{2\cdot 4\cdot 6\cdots(2n+2)}{1\cdot 4\cdot 7\cdots(3n+1)} z^n, \qquad \text{(ii)} \sum_{n\geqslant 1} \frac{z^{3n}}{n2^n}, \qquad \text{(iii)} \sum_{n\geqslant 0} \frac{n^n z^n}{n!}, \qquad \text{(iv)} \sum_{n\geqslant 1} n^{\sqrt{n}} z^n$$

8. In this question, you should only assume the properties of exponential and logarithm seen in lectures.

(a) By applying the mean value theorem to $\log(1+x)$ on $[0, a/n]$ with $n > |a|$, prove carefully that $(1+a/n)^n \to e^a$ as $n \to \infty$.

(b) Find $\lim_{n\to\infty} n(a^{1/n} - 1)$, where $a > 0$.

9. Recall the properties of sine and cosine from lectures. For $x \in \mathbb{R}$, set $\tan x = \frac{\sin x}{\cos x}$.

(a) Find the derivative of $\tan x$.

(b) Justify that there is a differentiable inverse function $\tan^{-1} x$ for $x \in \mathbb{R}$. What is its derivative?

(c) Now let $g(x) = x - x^3/3 + x^5/5 + \cdots$ for $|x| < 1$. By considering $g'(x)$, explain carefully why $\tan^{-1} x = g(x)$ for $|x| < 1$.

10. The *infinite product* $\prod_{n=1}^\infty (1 + a_n)$ is said to *converge* if the sequence of partial products $(p_N)_N$ with $p_N = (1+a_1)\cdots(1+a_N)$ converges.

(a) Suppose that $a_n \geqslant 0$ for all n. Putting $s_m = a_1 + \cdots + a_m$, prove that $s_N \leqslant p_N \leqslant e^{s_N}$, and deduce that $\prod_{n=1}^\infty (1+a_n)$ converges if and only if $\sum_{n=1}^\infty a_n$ converges.

(b) Evaluate $\prod_{n=2}^\infty (1 + 1/(n^2-1))$.

11. Let $f(x) = \log(1-x^2)$. [In what follows, the bounds $8x^2/3$ and $1/9n^2$ are not best possible; they are merely good enough for the conclusion.]

(a) Use the mean value theorem to show that $|f(x)| \leqslant 8x^2/3$ for $0 \leqslant x \leqslant 1/2$.

(b) Now let $I_n = \int_{n-1/2}^{n+1/2} \log x \, dx - \log n$ for $n \in \mathbb{N}$. Show that $I_n = \int_0^{1/2} f(t/n)\, dt$ and hence that $|I_n| \leqslant 1/9n^2$.

(c) By considering $\sum_{j=1}^n I_j$, deduce that $e^n n!/n^{n+1/2} \to \ell$ for some constant ℓ.

12. Let $I_n = \int_0^{\pi/2} \cos^n x$. Prove that $nI_n = (n-1)I_{n-2}$, and hence $\frac{2n}{2n+1} \leqslant \frac{I_{2n+1}}{I_{2n}} \leqslant 1$. Deduce Wallis's Product:

$$\frac{\pi}{2} = \lim_{n\to\infty} \frac{2\cdot 2\cdot 4\cdot 4\cdots 2n\cdot 2n}{1\cdot 3\cdot 3\cdot 5\cdots(2n-1)\cdot(2n+1)} = \lim_{n\to\infty} \frac{2^{4n}}{2n+1}\binom{2n}{n}^{-2}.$$

By taking note of the previous exercise, prove that $e^n n!/n^{n+1/2} \to \sqrt{2\pi}$ (Stirling's formula).

13. Let $I_n(\theta) = \int_{-1}^{1}(1-x^2)^n \cos(\theta x)dx$. Prove that

$$\theta^2 I_n = 2n(2n-1)I_{n-1} - 4n(n-1)I_{n-2}$$

for $n \geq 2$, and hence that

$$\theta^{2n+1} I_n(\theta) = n!(P_n(\theta)\sin\theta + Q_n(\theta)\cos\theta),$$

where P_n and Q_n are polynomials of degree at most $2n$ with integer coefficients. Deduce that π is irrational.

Email comments and suggestions to `rbdt2@cam.ac.uk`