ANALYSIS I—EXAMPLES 2

(updated 5 February 2026)

Exercises

1a. Show that $a \in X$ is an accumulation point of X if and only if there exists $(x_n) \in X \setminus \{a\}$ such that $x_n \to a$.

1b. Let $f\colon X \subset \mathbb{C} \to \mathbb{C}$. Recall that we say $f(x) \to y$ as $x \to a$ if

$$\forall \varepsilon > 0\, \exists \delta > 0 \text{ s.t. } |x-a| < \delta, x \in X \setminus \{a\} \implies |f(x) - y| < \epsilon.$$

For each $\varepsilon > 0$, let $D(\varepsilon) = \{\delta > 0\colon |x-a| < \delta, x \in X \setminus \{a\} \implies |f(x)-y| < \epsilon\}$. Set $\delta_{\max}(\varepsilon) = \sup D(\epsilon) \in (0,\infty)$ if $D(\epsilon)$ is bounded, and $\delta_{\max}(\varepsilon) = \infty$ if $D(\epsilon)$ is unbounded. Show that $\varepsilon_1 < \varepsilon_2 \implies \delta_{\max}(\varepsilon_1) \le \delta_{\max}(\varepsilon_2)$.

1c. Write down the definition of "$f(x) \to y$ as $x \to \infty$" for $y \in \mathbb{C}$. Prove that $f(x) \to y$ as $x \to \infty$ if, and only if, $f(x_n) \to y$ for every sequence such that $x_n \to \infty$. What if we replace y with ∞?

1d. Recall that we say f is continuous at a if

$$\forall \varepsilon > 0\, \exists \delta > 0 \text{ s.t. } |x-a| < \delta \implies |f(x) - f(a)| < \epsilon.$$

Give an example to show that there exist functions $g\colon \mathbb{R} \to \mathbb{R}$ for which there is $a \in \mathbb{R}$ such that

$$\forall \delta > 0\, \exists \varepsilon > 0 \text{ s.t. } |x-a| < \delta \implies |g(x) - g(a)| < \varepsilon,$$

yet whose graph cannot be sketched without lifting one's pencil.

1e. Let $f\colon [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Use the mean value theorem to show that

$$\begin{aligned} f'(x) \ge 0 \quad \forall\, x \in (a,b) &\implies f \text{ increasing on } [a,b]; \\ f'(x) \le 0 \quad \forall\, x \in (a,b) &\implies f \text{ decreasing on } [a,b]; \\ f'(x) = 0 \quad \forall\, x \in (a,b) &\implies f \text{ constant on } [a,b]. \end{aligned}$$

Problems

2. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x, & x \in \mathbb{Q} \\ 1-x, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}.$$

Identify the set of continuity points of f, $\{a : f \text{ is continuous at } a\}$.

3. Let $f, g : \mathbb{R} \to \mathbb{R}$ be such that $f(x) \to \ell$ as $x \to a$ and $g(y) \to k$ as $y \to \ell$. Must it be true that $g(f(x)) \to k$ as $x \to a$?

4. Let $f_n : [0,1] \to [0,1]$ be continuous, $n \in \mathbb{N}$.

(a) Let $h_n(x) = \max\{f_1(x), f_2(x), \ldots, f_n(x)\}$. Show that h_n is continuous on $[0,1]$ for each $n \in \mathbb{N}$.

(b) Must $h(x) = \sup\{f_n(x) : n \in \mathbb{N}\}$ be continuous?

5. ($\star$) Let $f : [a,b] \to \mathbb{R}$ be bounded. Suppose that $f((x+y)/2) \leqslant (f(x)+f(y))/2$ for all $x, y \in [a,b]$. Prove that f is continuous on (a,b). Must it be continuous at a and b too?

6. Let $f(x) = \sin^2 x + \sin^2(x + \cos^7 x)$. Assuming the familiar features of sin, cos without justification, prove that there exists $k > 0$ such that $f(x) \ge k$ for all $x \in \mathbb{R}$.

7. Suppose that $f : [0,1] \to \mathbb{R}$ is continuous, that $f(0) = f(1) = 0$, and that for every $x \in (0,1)$ there exists $0 < \delta < \min\{x, 1-x\}$ with $2f(x) = f(x-\delta) + f(x+\delta)$. Show that $f(x) = 0$ for all x.

8. Let $f : [0,1] \to [0,1]$ be a continuous function. Prove that there exists $c \in [0,1]$ such that $f(c) = c$. Such a c is called a *fixed point* of f. Give an example of a bijection of $[0,1]$ with no fixed point. If $h : (0,1) \to (0,1)$ is a continuous bijection, must it have a fixed point?

9. ($\star\star$) Construct a function $f : \mathbb{R} \to \mathbb{R}$ that takes every value on every interval – in other words for every $a < b$ and every c there is an x with $a < x < b$ such that $f(x) = c$.

10. Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Which of the following must be true?

(a) If f is increasing then $f'(x) \geqslant 0$ for all $x \in (a,b)$.
(b) If $f'(x) \geqslant 0$ for all $x \in (a,b)$ then f is increasing.
(c) If f is strictly increasing then $f'(x) > 0$ for all $x \in (a,b)$.
(d) If $f'(x) > 0$ for all $x \in (a,b)$ then f is strictly increasing.

11. Let $f(x) = x + 2x^2 \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Show that f is differentiable everywhere and that $f'(0) = 1$, but that there is no interval around 0 on which f is increasing.

12. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable for all x. Prove that

$$\lim_{x\to\infty} f'(x) = \ell \implies \lim_{x\to\infty} \frac{f(x)}{x} = \ell.$$

If $f(x)/x \to \ell$ as $x \to \infty$, must $f'(x)$ tend to a limit?

13. Prove that $2x^5 + 3x^4 + 2x + 16 = 0$ has no real solutions outside $[-2,-1]$ and exactly one inside.