ANALYSIS I—EXAMPLES 1

(updated 24 January 2025)

Exercises

1a. Suppose $(a_n), (b_n)$ are two sequences of real numbers. Prove that if $a_n \to a$ and $b_n \to b$ then $a_n + b_n \to a + b$.

1b. Prove the Bolzano-Weierstrass theorem in $\mathbb{C}$: if (z_n) is a bounded complex sequence, it has at least one convergent subsequence. *[Hint: $z_n = x_n + iy_n$ where (x_n) and (y_n) are bounded real sequences.]*

1c. If (x_n) satisfies $|x_n - x_{n+1}| \to 0$ as $n \to \infty$, must (x_n) be Cauchy?

1d. If $\sum a_n$ and $\sum b_n$ converge, must $\sum a_n b_n$ converge?

Problems

2. Sketch the graphs of $y = x$ and $y = (x^4+1)/3$, and thereby illustrate the behaviour of the real sequence a_n where $a_{n+1} = (a_n^4 + 1)/3$. For which of the three starting cases $a_1 = 0$, $a_1 = 1$ and $a_1 = 2$ does the sequence converge? Now prove your assertion.

3. Let $a_1 > b_1 > 0$ and let $a_{n+1} = (a_n + b_n)/2$, $b_{n+1} = 2a_n b_n/(a_n + b_n)$ for $n \geqslant 1$. Show that $a_n > a_{n+1} > b_{n+1} > b_n$ and deduce that the two sequences converge to a common limit. What limit?

4. ($\star$) Can we write the open interval (0,1) as a disjoint union of closed intervals of positive length?

5. Prove that every real sequence has a monotonic subsequence. Deduce the Bolzano-Weierstrass theorem.

6. Let (x_n) be a bounded real sequence that does not converge. Prove that it has two convergent subsequences with different limits.

7. Suppose that $\sum a_n$ diverges to infinity and $a_n > 0$. Construct a sequence (b_n) with $b_n/a_n \to 0$ and $\sum b_n$ divergent.

8. Let $z \in \mathbb{C}$ such that $z^{2^j} \neq 1$ for any positive integer j. Show that the series

$$\frac{z}{1-z^2} + \frac{z^2}{1-z^4} + \frac{z^4}{1-z^8} + \frac{z^8}{1-z^{16}} + \cdots$$

converges to $z/(1-z)$ if $|z| < 1$, converges to $1/(1-z)$ if $|z| > 1$, and diverges if $|z| = 1$.

9. Investigate the convergence of the following series. For those expressions containing the complex number z, find those z for which convergence occurs.

$$\sum_n \frac{\sin n}{n^2} \qquad \sum_n \frac{n^2 z^n}{5^n} \qquad \sum_n \frac{(-1)^n}{4+\sqrt{n}} \qquad \sum_n \frac{z^n(1-z)}{n}$$

10. Show that $\sum 1/[n(\log n)^\alpha]$ converges if $\alpha > 1$ and diverges otherwise. Does $\sum 1/(n \log n \log\log n)$ converge?

11. For $n \geqslant 1$, let

$$a_n = \frac{1}{\sqrt{n}} + \frac{(-1)^{n-1}}{n}.$$

Show that each a_n is positive and that $\lim a_n = 0$. Show also that $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ diverges.

12. Let a_n and b_n be two sequences and let $S_n = \sum_{j=1}^{n} a_j$ with the convention $S_0 = 0$.

(a) Show that for any $1 \leqslant m \leqslant n$ we have:

$$\sum_{j=m}^{n} a_j b_j = S_n b_n - S_{m-1} b_m + \sum_{j=m}^{n-1} S_j (b_j - b_{j+1}).$$

(b) Suppose now that b_n is a decreasing sequence of positive terms tending to zero and that S_n is a bounded sequence. Prove that $\sum_{j=1}^{\infty} a_j b_j$ converges.

(c) Does the series $\sum_{n=1}^{\infty} \frac{\cos(n)}{n}$ converge or diverge?

(d) Deduce the alternating series test from the assertion in part (b).

13. Consider the two series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots \qquad \text{and} \qquad 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots,$$

having the same terms but taken in a different order. Let s_n and t_n be the corresponding partial sums up to n terms.

(a) Show that $s_{2n} = h_{2n} - h_n$ and $t_{3n} = h_{4n} - \frac{1}{2} h_{2n} - \frac{1}{2} h_n$, where $h_n = \sum_{k \leq n} \frac{1}{k}$.

(b) Show that s_n converges to a limit $s > 0$ and that t_n converges to $3s/2$.

14. Let (x_n) and (y_n) be real sequences with $x_n \to 0$ and $y_n \to 0$ as $n \to \infty$.

(a) Show that there is a sequence (σ_n) of signs (i.e. $\sigma_n \in \{-1, +1\}$ for all n) such that $\sum \sigma_n x_n$ is convergent.

(b) ($\star\star$) Must there be a sequence (σ_n) of signs such that $\sum \sigma_n x_n$ and $\sum \sigma_n y_n$ are both convergent?

Notes: The questions are ordered according to the order of relevant lectured material; challenging questions are denoted by ($\star$) and *very* challenging questions by ($\star\star$).

Email comments and suggestions to rbdt2@cam.ac.uk