Analysis I **Lent term 2020**

Example Sheet 4

1. Give an example of an integrable function $f : [0,1] \to \mathbb{R}$ with $f \geq 0$, $\int_0^1 f(x)dx = 0$, and $f(x) > 0$ for some $x \in [0,1]$. Show that this cannot happen if f is continuous.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be monotonic. Show that $\{x \in \mathbb{R} \,|\, f \text{ is discontinuous at } x\}$ is countable. Let $x_n, n \geq 1$ be a sequence of distinct points in $(0,1)$ and define $f_n(x) = 0$ if $0 \leq x < x_n$, $f_n(x) = 1$ otherwise. Define $f(x) = \sum_{n=1}^{\infty} 2^{-n} f_n(x)$. Show that this series converges for all $x \in [0,1]$, and that f is integrable. Show that f is discontinuous at every x_n.

3. Define $f : [0,1] \to \mathbb{R}$ by $f(p/q) = 1/q$, where $p, q \in \mathbb{N}$ are relatively prime, and $f(x) = 0$ if x is irrational. Show that f is integrable. What is $\int_0^1 f(x)dx$?

4. Give an example of a continuous function $f : [0,\infty) \to [0,\infty)$ such that $\int_0^\infty f(x)dx$ exists, but f is unbounded.

5. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is C^1, $f(0) = 0$, and $|f'(x)| \leq M$ for $x \in [0,1]$. Show that $|\int_0^1 f(x)dx| \leq M/2$. If in addition $f(1) = 0$, show that $|\int_0^1 f(x)dx| \leq M/4$. What can you say if $f(0) = 0$ and $|f'(x)| \leq kx$ for some $k \in \mathbb{R}$?

6. Let $f : [0,1] \to \mathbb{R}$ be continuous. Let $G(x,t) = t(x-1)$ for $t \leq x$ and $G(x,t) = x(t-1)$ for $t \geq x$. Let $g(x) = \int_0^1 f(t)G(x,t)dt$. Show that $g''(x)$ exists for $x \in (0,1)$ and is equal to $f(x)$.

7. Determine whether the following improper integrals converge:

   (a) $\int_1^\infty \sin^2(1/x)dx$

   (b) $\int_0^\infty x^p \exp(-x^q)dx$ for $p, q > 0$

   (c) $\int_0^\infty \sin(x^2)dx$

8. Show that $\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} \to \log 2$ as $n \to \infty$. What is $\lim_{n\to\infty} \frac{1}{n+1} - \frac{1}{n+2} + \ldots + \frac{(-1)^{n-1}}{2n}$?

9. Let $f(x) = \log(1 - x^2)$. Use the mean value theorem to show that $|f(x)| \leq 8x^2/3$ for $x \in [0, 1/2]$. Now let

$$I_n = \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \log x \, dx - \log n$$

for $n \in \mathbb{N}$. Show that $I_n = \int_0^{1/2} f(t/n)dt$ and hence that $|I_n| \leq 1/(9n^2)$. By considering $\sum_{j=1}^n I_j$, show that the sequence $(n! e^n n^{-n-1/2})$ converges. (The bounds

$8x^2/3$ and $1/(9n^2)$ are not the best possible; they are merely good enough for the conclusion.)

10. Let $I_n = \int_0^{\pi/2} \cos^n x\, dx$. Prove that $nI_n = (n-1)I_{n-2}$ and hence $\frac{2n}{2n+1} \leq \frac{I_{2n+1}}{I_{2n}} \leq 1$. Deduce Wallis's product formula:

$$\frac{\pi}{2} = \lim_{n\to\infty} \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot \cdots \cdot 2n \cdot 2n}{1 \cdot 3 \cdot 3 \cdot 5 \cdots (2n-1) \cdot (2n+1)} = \lim_{n\to\infty} \frac{2^{4n}}{2n+1} \binom{2n}{n}^{-2}.$$

Using the previous exercise, prove that $n! e^n n^{-n-1/2} \to \sqrt{2\pi}$ (Stirling's formula).

11. Let $I_n(\theta) = \int_{-1}^{1} (1-x^2)^n \cos(\theta x) dx$. Prove that $\theta^2 I_n = 2n(2n-1)I_{n-1} - 4n(n-1)I_{n-2}$ for $n \geq 2$, and hence that $\theta^{2n+1} I_n(\theta) = n!(P_n(\theta)\sin\theta + Q_n(\theta)\cos\theta)$, where P_n and Q_n are polynomials of degree $\leq 2n$ with integer coefficients. Deduce that π is irrational.

12. A function $g : [a,b] \to \mathbb{R}$ is said to have *bounded variation* if there is a constant K such that whenever $a_0 < a_1 \cdots < a_n$ is a dissection of $[a,b]$, $\sum_{i=1}^{n} |g(a_i) - g(a_{i+1})| \leq K$. Show that if g has bounded variation, g is integrable. Show also that if $g = f_1 - f_2$, where f_1 and f_2 are both increasing, then g has bounded variation. Give an example of a continuous (hence integrable) function which does not have bounded variation.

13. Suppose that $f : [a,b] \to \mathbb{R}$ is integrable, that $f \geq 0$, and that $\int_a^b f(x)dx = 0$. Show that for every $\epsilon > 0$ and every closed interval $I \subset [a,b]$ of positive length, there is a closed interval $J \subset I$ such that J has positive length and $f(x) \leq \epsilon$ for all $x \in J$. Deduce that if $f > 0$, $\int_a^b f(x)dx > 0$.

14. Show that if $f : [a,b] \to \mathbb{R}$ is integrable, then f is continuous at infinitely many $x \in [a,b]$.

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