Analysis I **Lent term 2020**

Example Sheet 3

1. Show that $\lim_{x\to+\infty} x^n/\exp(x) = 0$ for any $n \in \mathbb{N}$ directly from the definition of the exponential function.

2. Show that $(1+\frac{a}{n})^n \to \exp(a)$ as $n \to \infty$ by applying the mean value theorem to $\log(1+x)$ on the interval $[0, \frac{a}{n}]$. Compare with Problem 7 on Example Sheet 1.

3. For $a > 0$, find $\lim_{n\to\infty} n(a^{1/n} - 1)$.

4. Find the flaw in the following argument: "Let f be differentiable on (a,b) and suppose that $c \in (a,b)$. If $c+h \in (a,b)$, then $(f(c+h) - f(c))/h = f'(c+\theta h)$ for some $\theta \in [0,1]$. Let $h \to 0$, then $f'(c+\theta h) \to f'(c)$. Thus f' is continuous at c."

5. Suppose that f is twice differentiable at x. Prove that
$$f''(x) = \lim_{h\to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}.$$
If $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable and satisfies $f(x) = x^2\alpha(x)$, where $\alpha(x) \to 0$ as $x \to 0$, show that $f'(0) = f''(0) = 0$.

6. We say $\lim_{x\to\infty} f(x) = c$ if for every $\epsilon > 0$ there exists M such that $|f(x) - c| < \epsilon$ whenever $x > M$. Suppose that $f, g : (0,\infty) \to \mathbb{R}$ are differentiable, that
$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0,$$
and that there is some N such that $g'(x) \neq 0$ for $x > N$. If $\lim_{x\to\infty} f'(x)/g'(x) = c$, show that $\lim_{x\to\infty} f(x)/g(x) = c$ as well.

7. Let $f(x) = \sqrt{x}$. Express $f(1+h)$ as a quadratic in h plus a remainder term involving h^3. By taking $h = -0.02$, find an approximate value for $\sqrt{2}$ and prove it is accurate to seven decimal places.

8. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \exp(-1/x^2)$ for $x \neq 0$, $f(0) = 0$. Prove carefully that f is infinitely differentiable, and that $f^{(k)}(0) = 0$ for all $k \in \mathbb{N}$. Hence the Taylor series of f centered at 0 does not converge to $f(x)$ for any $x \neq 0$. Explain how this fact is compatible with Taylor's theorem.

9. Find the radius of convergence of the following power series:
$$\sum_n \frac{2\cdot 4\cdot 6\cdots(2n+2)}{1\cdot 4\cdot 7\cdots(3n+1)} z^n \qquad \sum_n \frac{z^{3n}}{n2^n} \qquad \sum_n \frac{n^n z^n}{n!} \qquad \sum_n n^{\sqrt{n}} z^n.$$

10. Prove that $\tan : (-\pi/2, \pi/2) \to \mathbb{R}$ is a bijection. Now let $g(x) = x - x^3/3 + x^5/5 + \ldots$ for $|x| < 1$. By considering $g'(x)$, show that $\tan^{-1}(x) = g(x)$ for $|x| < 1$.

11. Show that $\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$. Use this identity to compute π to five decimal places. (Machin used it to compute the first 100.) Justify the accuracy of your calculation.

12. We say that $\prod_{n=1}^{\infty}(1 + a_n)$ converges if the sequence $p_n = (1 + a_1)(1 + a_2)\ldots(1 + a_n)$ converges. Suppose that $a_n \geq 0$ for all n. Putting $s_n = a_1 + a_2 + \ldots + a_n$, prove that $s_n \leq p_n \leq \exp(s_n)$. Deduce that $\prod_{n=1}^{\infty}(1 + a_n)$ converges if and only if $\sum_{n=1}^{\infty} a_n$ converges. Evaluate $\prod_{n=2}^{\infty}\left(1 + \frac{1}{n^2-1}\right)$.

13. (i) If $z \in \mathbb{C} \setminus \{0\}$, prove that there exists $\lambda \in \mathbb{C}$ such that $\exp(\lambda) = z$. (ii) Let $L(z) = \sum_{n=1}^{\infty} \frac{-1}{n}(1 - z)^n$. Prove that L is well-defined on $D = \{z \in \mathbb{C} \mid |1 - z| < 1\}$, and that $L : D \to \mathbb{C}$ is complex differentiable. What is its derivative? By considering the function $z\exp(-L(z))$, show that $\exp(L(z)) = z$ for all $z \in D$. (iii) Show that there is no continuous function $L : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ satisfying $\exp(L(z)) = z$ for all $z \in \mathbb{C}$.

14. Construct a C^∞ function $f : \mathbb{R} \to \mathbb{R}$ which satisfies $f(x) = 0$ for $x \leq 0$ and $f(x) = 1$ for $x \geq 1$. Deduce that if $g_1, g_2 : \mathbb{R} \to \mathbb{R}$ are C^∞ and $a < b$, then there is a C^∞ function $g : \mathbb{R} \to \mathbb{R}$ which satisfies $g(x) = g_1(x)$ for $x \leq a$ and $g(x) = g_2(x)$ for $x \geq b$.

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