Analysis I **Lent term 2019**

Example Sheet 2

1. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$ for $x \in \mathbb{Q}$ and $f(x) = -x^2$ for $x \notin \mathbb{Q}$. At which points is f continuous? Differentiable?

2. Prove that the absolute value function $|\cdot| : \mathbb{C} \to \mathbb{R}$ is continuous. (You should work directly from the ϵ-δ definition of continuity; do not use continuity of the square root function.)

3. Suppose that $f_k : [0,1] \to [0,1]$ is continuous for each $k \in \mathbb{N}$. Let $h_n(x) = \max\{f_1(x), f_2(x), \ldots, f_n(x)\}$. Show that h_n is continuous. Now suppose that there is a constant M such that $f_k(x) \leq M$ for all $k \in \mathbb{N}$ and $x \in [0,1]$. Is the function h defined by $h(x) = \sup\{f_k(x) \,|\, k \in \mathbb{N}\}$ necessarily continuous?

4. Define a function $f : \mathbb{R} \to \mathbb{R}$ as follows. If x is irrational, then $f(x) = 0$, while if f is rational, then $f(x) = 1/q$, where q is the denominator of x. (That is, $x = p/q$, where p and q are coprime integers and $q > 0$.) Prove that f is continuous at every irrational and discontinuous at every rational.

5. Let $f : [0,1], \to [0,1]$ be a continuous function. Prove that there exists some $c \in [0,1]$ with $f(c) = c$. Such a c is called a *fixed point* of f. Give an example of a bijection $g : [0,1] \to [0,1]$ with no fixed point. If $h : (0,1) \to (0,1)$ is a continuous bijection, must it have a fixed point?

6. Suppose $f : [a,b] \to \mathbb{R}$ is *strictly increasing*; that is $f(x) < f(y)$ whenever $x < y$. Show that f is continuous if and only if $f([a,b]) = [f(a), f(b)]$.

7. Let I be an interval and suppose that $f : I \to \mathbb{R}$ is continuous and injective. Show that $f^{-1} : f(I) \to I$ is continuous.

8. Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies $|f(x) - f(y)| \leq |x - y|^2$ for all $x, y, \in \mathbb{R}$. Show that f is constant.

9. Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Which of the following statements are always true and which are sometimes false?

   (a) If f is increasing, then $f'(x) \geq 0$ for every $x \in (a,b)$.

   (b) If $f'(x) \geq 0$ for every $x \in (a,b)$, then f is increasing.

   (c) If f is strictly increasing, then $f'(x) > 0$ for every $x \in (a,b)$.

   (d) If $f'(x) > 0$ for every $x \in (a,b)$, then f is strictly increasing.

10. Prove that the equation $2x^5 + 3x^4 + 2x + 16 = 0$ has exactly one real root, and that this root is in the interval $[-2, -1]$.

11. (i) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x + 2x^2 \sin(1/x)$ if $x \neq 0$ and $f(0) = 0$. Show that f is differentiable and that $f'(0) = 1$, but there is no interval around 0 on which f is increasing. (You may assume standard facts about $\sin x$, $\cos x$ and their derivatives.) (ii) Give an example of a differentiable function $g : \mathbb{R} \to \mathbb{R}$ with the property that g' is not bounded on the interval $(-\delta, \delta)$ for any $\delta > 0$.

12. Suppose $f : [a, b] \to \mathbb{R}$ is differentiable, and that $f'(a) < k < f'(b)$. Show that there is some $c \in (a, b)$ with $f'(c) = k$. (Hint: consider $g(x) = f(x) - kx$.)

13. Let $g : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $g(0) = g'(0) = 0$ and $g''(0)$ exists and is positive. Prove there exists $x > 0$ such that $g(x) > 0$. Now let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f(0) = 0$ and $f''(0)$ exists and is positive. Prove there is some $x > 0$ such that $f(2x) > 2f(x)$.

14. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable, and that $f'(x) \to c$ as $x \to \infty$. Show that $f(x)/x \to c$ as $x \to \infty$. If $f(x)/x \to c$ as $x \to \infty$, must $f'(x) \to c$ as $x \to \infty$?

15. * Find a function $f : \mathbb{R} \to \mathbb{R}$ which takes every real value on every open interval. (In other words $f((a, b)) = \mathbb{R}$ for any $a < b$.)

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