1. Using the fact that log is differentiable at 1, prove that $\left(1+\frac{a}{n}\right)^n \to \exp(a)$ as $n \to \infty$ for every $a \in \mathbb{R}$. Deduce that $\exp(z) = \mathrm{e}^z$ for every $z \in \mathbb{C}$.

2. (i) Let $g\colon \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $g(0) = g'(0) = 0$ and $g''(0)$ exists and is positive. Prove that there exists $x > 0$ such that $g(x) > 0$.
(ii) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f(0) = 0$ and $f''(0)$ exists and is positive. Prove that there exists $x > 0$ such that $f(2x) > 2f(x)$.

3. Prove Cauchy's mean value theorem: let $f, g\colon [a,b] \to \mathbb{R}$ be continuous functions which are differentiable on the open interval (a,b); show that for some $c \in (a,b)$ the vectors $\big(f(b)-f(a), g(b)-g(a)\big)$ and $\big(f'(c), g'(c)\big)$ in $\mathbb{R}^2$ are parallel. Does this generalize to three or more functions?

4. Let $f\colon \mathbb{R} \to \mathbb{R}$ be differentiable everywhere. Prove that if $f'(x) \to \ell$ as $x \to \infty$ then $f(x)/x \to \ell$ as $x \to \infty$. If $f(x)/x \to \ell$ as $x \to \infty$, does it follow that $f'(x) \to \ell$?

5. Define $f\colon \mathbb{R} \to \mathbb{R}$ by letting $f(x) = \mathrm{e}^{-1/x^2}$ for $x \neq 0$ and $f(0) = 0$. Show that f is infinitely differentiable and find its Taylor series at 0.

6. Show that $\tan x = \frac{\sin x}{\cos x}$ defines a bijection from $(-\frac{\pi}{2}, \frac{\pi}{2})$ onto $\mathbb{R}$. Prove that the inverse function arctan is differentiable and find its derivative. Why is it reasonable to guess that $\arctan x = x - x^3/3 + x^5/5 - \dots$ when $|x| < 1$? Verify this guess by considering derivatives.

7. Find the radius of convergence of each of the following power series.

$$\sum_{n=0}^{\infty} \frac{2\cdot 4\cdot 6 \dots (2n+2)}{1\cdot 4\cdot 7 \dots (3n+1)} z^n \qquad \sum_{n=1}^{\infty} \frac{z^{3n}}{n2^n} \qquad \sum_{n=0}^{\infty} \frac{n^n z^n}{n!} \qquad \sum_{n=1}^{\infty} n^{\sqrt{n}} z^n$$

8. We say that a function $f\colon \mathbb{R} \to \mathbb{R}$ has a *local maximum at* a if for some $r > 0$, we have $f(x) \leqslant f(a)$ for all $x \in (a-r, a+r)$. A *local minimum* is defined similarly. Assuming that f is differentiable at a, prove that if f has a local maximum or minimum at a then $f'(a) = 0$, but that the converse fails in general. However, show that if f is twice differentiable at a, $f'(a) = 0$ and $f''(a) < 0$ (or $f''(a) > 0$), then f has a local maximum (respectively, minimum) at a.

9. Assume that f is twice differentiable at x. Prove that

$$f''(x) = \lim_{h\to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} .$$

10. Let f be continuous on $[-1, 1]$ and twice differentiable on $(-1, 1)$. Let $\varphi(x) = (f(x) - f(0))/x$ for $x \neq 0$ and $\varphi(0) = f'(0)$. Show that φ is continuous on $[-1, 1]$ and differentiable on $(-1, 1)$. By using a second-order mean value theorem for f, show that $\varphi'(x) = f''(\theta x)/2$ for some $\theta \in (0, 1)$. Hence prove that there exists $c \in (-1, 1)$ such that $f''(c) = f(-1) + f(1) - 2f(0)$.

11. Let $f\colon I \to \mathbb{R}$ be a differentiable function on the open interval I. Show that if $f'(a) < y < f'(b)$ for some $a < b$ in I and $y \in \mathbb{R}$, then there exists $x \in I$ with $a < x < b$ and $f'(x) = y$. [*Note that f' is not assumed to be continuous.*] Deduce that if $f'(x) \neq 0$ for all $x \in I$, then f is strictly monotonic.

12. (i) Let $z \in \mathbb{C}\setminus\{0\}$. We say that $\varphi \in \mathbb{R}$ is a *choice of argument of* z if $\mathrm{e}^{\mathrm{i}\varphi} = z/|z|$, and we denote by $\arg z$ the set of all such $\varphi \in \mathbb{R}$. Show that $\arg z$ contains a unique element $\theta \in [0, 2\pi)$, and then $\arg(z) = \{\theta + 2\pi n : n \in \mathbb{Z}\}$.
(ii) Show that there is no continuous choice of argument on $\mathbb{C}\setminus\{0\}$, *i.e.*, there is no continuous function $\theta\colon \mathbb{C}\setminus\{0\} \to \mathbb{R}$ such that $\theta(z) \in \arg z$ for all $z \in \mathbb{C}\setminus\{0\}$. [*Hint: assume such θ exists, and consider the function $f(x) = \frac{1}{\pi}\big(\theta(\mathrm{e}^{\mathrm{i}x}) - \theta(\mathrm{e}^{\mathrm{i}x+\mathrm{i}\pi})\big)$.*]

13. (i) Let $z \in \mathbb{C}\setminus\{0\}$. Show that there exists $\lambda \in \mathbb{C}$ such that $\mathrm{e}^{\lambda} = z$. Such a λ is called a *choice of logarithm of z*.
(ii) Show that the power series $\sum_{n=1}^{\infty} \frac{-1}{n}(1-z)^n$ has radius of convergence 1. Let $D = \{z \in \mathbb{C} : |z-1| < 1\}$, and define $L\colon D \to \mathbb{C}$ by $L(z) = \sum_{n=1}^{\infty} \frac{-1}{n}(1-z)^n$. Show that L is complex differentiable and find its derivative. By considering the function $f(z) = z\mathrm{e}^{-L(z)}$, show that $L(z)$ is a choice of logarithm of z for every $z \in D$.

14. (i) The *extended real line* is the set $\mathbb{R}^* = \mathbb{R} \cup \{-\infty, \infty\}$. The linear order of $\mathbb{R}$ is extended to $\mathbb{R}^*$ by declaring $-\infty < x < \infty$ for all $x \in \mathbb{R}$. Prove that in $\mathbb{R}^*$ every non-empty set has a supremum and an infimum, and that every monotonic sequence converges.

Let (x_n) be a sequence in $\mathbb{R}^*$. We define

$$\liminf x_n = \lim_{n\to\infty} \inf\{x_m : m \geqslant n\} \quad \text{and} \quad \limsup x_n = \lim_{n\to\infty} \sup\{x_m : m \geqslant n\}\ .$$

Show that $\liminf x_n \leqslant \limsup x_n$ with equality if and only if (x_n) converges in $\mathbb{R}^*$, and then $\lim x_n$ is their common value.
(ii) Show that the power series $\sum a_n z^n$ has radius of convergence R given by

$$R = \frac{1}{\limsup |a_n|^{\frac{1}{n}}}$$

where we define $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$.