1. Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ satisfies the inequality $|f(x) - f(y)| \leqslant |x - y|^2$ for every $x, y \in \mathbb{R}$. Show that f is constant.

2. (i) Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2 \sin\left(\frac{1}{x}\right)$ if $x \neq 0$ and $f(0) = 0$. Prove that f is differentiable everywhere. For which x is f' continuous at x?
(ii) Give an example of a function $g\colon \mathbb{R} \to \mathbb{R}$ that is differentiable everywhere such that g' is not bounded on the interval $(-\delta, \delta)$ for any $\delta > 0$.

3. Construct a function $f\colon \mathbb{R} \to \mathbb{R}$ which is differentiable at 0 and discontinuous at every $x \neq 0$.

4. Using the fact that log is differentiable at 1, prove that $\left(1 + \frac{a}{n}\right)^n \to \exp(a)$ as $n \to \infty$ for every $a \in \mathbb{R}$. Deduce that $\exp(z) = \mathrm{e}^z$ for every $z \in \mathbb{C}$.

5. Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be differentiable functions with $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} g(x) = 0$. Assume that for some $c > 0$, we have $g'(x) \neq 0$ for all $x > c$ and that $\frac{f'(x)}{g'(x)} \to \ell$ as $x \to \infty$. Deduce that $g(x) \neq 0$ for all $x > c$ and that $\frac{f(x)}{g(x)} \to \ell$ as $x \to \infty$.

6. Define $f\colon \mathbb{R} \to \mathbb{R}$ by letting $f(x) = \mathrm{e}^{-1/x^2}$ for $x \neq 0$ and $f(0) = 0$. Show that f is infinitely differentiable and find its Taylor series at 0.

7. Show that $\tan x = \frac{\sin x}{\cos x}$ defines a bijection from $(-\frac{\pi}{2}, \frac{\pi}{2})$ onto $\mathbb{R}$. Prove that the inverse function arctan is differentiable and find its derivative. Why is it reasonable to guess that $\arctan x = x - x^3/3 + x^5/5 - \ldots$ when $|x| < 1$? Verify this guess by considering derivatives.

8. Find the radius of convergence of each of the following power series.

$$\sum_{n=0}^{\infty} \frac{2 \cdot 4 \cdot 6 \ldots (2n+2)}{1 \cdot 4 \cdot 7 \ldots (3n+1)} z^n \qquad \sum_{n=1}^{\infty} \frac{z^{3n}}{n 2^n} \qquad \sum_{n=0}^{\infty} \frac{n^n z^n}{n!} \qquad \sum_{n=1}^{\infty} n^{\sqrt{n}} z^n$$

9. We say that a function $f\colon \mathbb{R} \to \mathbb{R}$ has a *local maximum at a* if for some $r > 0$, we have $f(x) \leqslant f(a)$ for all $x \in (a-r, a+r)$. A *local minimum* is defined similarly. Assuming that f is differentiable at a, prove that if f has a local maximum or minimum at a then $f'(a) = 0$, but that the converse does not hold in general. However, show that if f is twice differentiable at a, $f'(a) = 0$ and $f''(a) < 0$ (or $f''(a) > 0$), then f has a local maximum (respectively, minimum) at a.

10. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function such that $\frac{f(x)}{x^{3/4}} \to \ell$ as $x \to \infty$. Show that $\sqrt{x + f(x)} - \sqrt{x} - \frac{1}{2}\frac{f(x)}{\sqrt{x}} \to -\frac{\ell^2}{8}$ as $x \to \infty$.

11. Let $f\colon I \to \mathbb{R}$ be a differentiable function on the open interval I. Show that if $f'(a) < y < f'(b)$ for some $a < b$ in I and $y \in \mathbb{R}$, then there exists $x \in I$ with $a < x < b$ and $f'(x) = y$. [*Note that f' is not assumed to be continuous.*] Deduce that if $f'(x) \neq 0$ for all $x \in I$, then f is strictly monotonic.

12. The infinite product $\prod_{n=1}^{\infty}(1 + x_n)$ is said to converge to x if the sequence of partial products $P_n = (1 + x_1)\dots(1 + x_n)$ converges to x. Suppose that $x_n \geqslant 0$ for every n. Write $S_n = x_1 + \dots + x_n$. Prove that $S_n \leqslant P_n \leqslant \mathrm{e}^{S_n}$ for every n, and deduce that $\prod_{n=1}^{\infty}(1 + x_n)$ converges if and only if $\sum_{n=1}^{\infty} x_n$ converges. Evaluate the product $\prod_{n=2}^{\infty}(1 + 1/(n^2 - 1))$.

13. (i) Let $z \in \mathbb{C} \setminus \{0\}$. We say that $\varphi \in \mathbb{R}$ is a *choice of argument of z* if $\mathrm{e}^{\mathrm{i}\varphi} = z/|z|$, and we denote by $\arg z$ the set of all such $\varphi \in \mathbb{R}$. Show that $\arg z$ contains a unique element $\theta \in [0, 2\pi)$, and then $\arg(z) = \{\theta + 2\pi n : n \in \mathbb{Z}\}$.
(ii) Show that there is no continuous choice of argument on $\mathbb{C} \setminus \{0\}$, *i.e.*, there is no continuous function $\theta\colon \mathbb{C} \setminus \{0\} \to \mathbb{R}$ such that $\theta(z) \in \arg z$ for all $z \in \mathbb{C} \setminus \{0\}$. [*Hint: assume such θ exists, and consider the function $f(x) = \frac{1}{\pi}\big(\theta(\mathrm{e}^{\mathrm{i}x}) - \theta(\mathrm{e}^{\mathrm{i}x + \mathrm{i}\pi})\big)$ for $0 \leqslant x \leqslant \pi$.*]

14. (i) Let $z \in \mathbb{C} \setminus \{0\}$. Show that there exists $\lambda \in \mathbb{C}$ such that $\mathrm{e}^{\lambda} = z$. Such a λ is called a *choice of logarithm of z*.
(ii) Show that the power series $\sum_{n=1}^{\infty} \frac{-1}{n}(1 - z)^n$ has radius of convergence 1. Let $D = \{z \in \mathbb{C} : |z - 1| < 1\}$, and define $L\colon D \to \mathbb{C}$ by $L(z) = \sum_{n=1}^{\infty} \frac{-1}{n}(1 - z)^n$. Show that L is complex differentiable and find its derivative. By considering the function $f(z) = z\mathrm{e}^{-L(z)}$, show that $L(z)$ is a choice of logarithm of z for every $z \in D$.

15. (i) Given a sequence (x_n) in $\mathbb{R}$, for each $m \in \mathbb{N}$ let $a_m = \inf\{x_n : n \geqslant m\}$ and $b_m = \sup\{x_n : n \geqslant m\}$. Then (a_m) is an increasing sequence in $\mathbb{R} \cup \{-\infty\}$, and hence tends to some element of $\mathbb{R} \cup \{-\infty, \infty\}$, which we denote by $\liminf x_n$. Similarly, (b_m) is a decreasing sequence in $\mathbb{R} \cup \{\infty\}$, and hence tends to some element of $\mathbb{R} \cup \{-\infty, \infty\}$, which we denote by $\limsup x_n$.

Show that $\liminf x_n \leqslant \limsup x_n$ with equality if and only if (x_n) converges in $\mathbb{R} \cup \{-\infty, \infty\}$, and then $\lim x_n$ is their common value.
(ii) Show that the power series $\sum a_n z^n$ has radius of convergence $R = \frac{1}{\limsup |a_n|^{\frac{1}{n}}}$.