1. Let $\sum_{n\geqslant 0} x_n$ and $\sum_{n\geqslant 0} y_n$ be absolutely convergent series. For each $n \geqslant 0$ let $z_n = \sum_{k=0}^{n} x_k y_{n-k}$. Prove that $\sum_{n\geqslant 0} z_n$ converges absolutely and that $\sum_{n=0}^{\infty} z_n = \sum_{n=0}^{\infty} x_n \cdot \sum_{n=0}^{\infty} y_n$.

2. Let (x_n) be a decreasing real sequence converging to zero, and let $\sum z_n$ be a real or complex series whose sequence of partial sums is bounded. Show that $\sum x_n z_n$ converges.

3. For each $n \in \mathbb{N}$ let $f_n \colon [0,1] \to [0,1]$ be a continuous function, and for each $n \in \mathbb{N}$ let h_n be defined by $h_n(x) = \max\{f_1(x), f_2(x), \ldots, f_n(x)\}$. Show that for each n the function h_n is continuous on $[0,1]$. Must the function h defined by $h(x) = \sup\{f_n(x) : n \in \mathbb{N}\}$ be continuous on $[0,1]$?

4. Let f be a function on a set A and let $a \in A$. Assume that for every sequence (x_n) in A that converges to a, the sequence $\big(f(x_n)\big)$ is convergent. Does it follow that f is continuous at a?

5. Let f be a function on a set A and let $B \subset A$. Assume that g is a continuous function on A that agrees with f on B. For which $x \in B$ can we deduce that f is continuous at x?

6. Let $g \colon [0,1] \to [0,1]$ be a continuous function. Prove that there exists some $c \in [0,1]$ such that $g(c) = c$. Such a c is called a *fixed point* of g.
Give an example of a bijection $h \colon [0,1] \to [0,1]$ with no fixed point. Give an example of a continuous bijection $p \colon (0,1) \to (0,1)$ with no fixed point.

7. Prove that the real polynomial $p(x) = 2x^5 + 3x^4 + 2x + 16$ takes the value 0 exactly once, and that the number where it takes that value is somewhere in the interval $[-2,-1]$.

8. A function f defined on a set A is *locally bounded* if every point in A has a neighbourhood on which f is bounded: for all $a \in A$ there exists $\delta > 0$ and $C \in \mathbb{R}$ such that if $x \in A$ and $|x - a| < \delta$ then $|f(x)| \leqslant C$. Show that every continuous function is locally bounded. Is the converse true? Show that a locally bounded function on a closed bounded interval is bounded.

9. Let $f\colon [0,1] \to \mathbb{R}$ be continuous with $f(0) = f(1) = 0$. Suppose that for every $x \in (0,1)$ there exists $\delta > 0$ such that both $x - \delta$ and $x + \delta$ belong to $(0,1)$ and $f(x) = \frac{1}{2}\big(f(x-\delta) + f(x+\delta)\big)$. Prove that $f(x) = 0$ for all $x \in [0,1]$.

10. Define a function $f\colon \mathbb{R} \to \mathbb{R}$ by setting $f(x) = 0$ if x is irrational, and $f(x) = 1/q$ when $x = p/q$ for coprime integers p and q with $q > 0$. Prove that f is continuous at every irrational and discontinuous at every rational. $^+$ Does there exist a function $g\colon \mathbb{R} \to \mathbb{R}$ which is continuous at every rational and discontinuous at every irrational?

11. Let I be an interval and $f\colon I \to \mathbb{R}$ be a continuous, injective function. Show that $f^{-1}\colon f(I) \to I$ is continuous.

12. (i) Let $g\colon \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $g(0) = g'(0) = 0$ and $g''(0)$ exists and is positive. Prove that there exists $x > 0$ such that $g(x) > 0$.
(ii) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f(0) = 0$ and $f''(0)$ exists and is positive. Prove that there exists $x > 0$ such that $f(2x) > 2f(x)$.

13. Let $f\colon \mathbb{R} \to \mathbb{R}$ be differentiable everywhere. Prove that if $f'(x) \to \ell$ as $x \to \infty$ then $f(x)/x \to \ell$ as $x \to \infty$. If $f(x)/x \to \ell$ as $x \to \infty$, does it follow that $f'(x) \to \ell$?

14. Prove Cauchy's mean value theorem: let $f, g\colon [a,b] \to \mathbb{R}$ be continuous functions which are differentiable on the open interval (a,b); show that for some $c \in (a,b)$ the vectors $\big(f(b), g(b)\big) - \big(f(a), g(a)\big)$ and $\big(f'(c), g'(c)\big)$ in $\mathbb{R}^2$ are parallel. Does this generalize to three or more functions?

15. A function $f\colon I \to \mathbb{R}$ on an interval I is *convex* if

$$f\big((1-t)x + ty\big) \leqslant (1-t)f(x) + tf(y) \qquad \forall\, x, y \in I\ \forall\, t \in [0,1]\ .$$

Assume now that I is an open interval. Show the following.
(i) If f is convex then it is continuous.
(ii) If f is convex then for each $c \in I$ there exists $m \in \mathbb{R}$ such that

$$m(x-c) + f(c) \leqslant f(x) \qquad \text{for all } x \in I\ ,$$

and if in addition f is differentiable at c then $f'(c)$ is the unique m that works. In general, must m be unique?
(iii) If f is twice differentiable and $f'' \geqslant 0$ on I, then f is convex.