1. Let (x_n) be a real sequence.

(i) Show that $x_n \to -\infty$ if and only if $-x_n \to \infty$.

(ii) Show that if $x_n \neq 0$ for all n and $x_n \to \infty$ then $\frac{1}{x_n} \to 0$.

(iii) If $x_n \neq 0$ for all n and $\frac{1}{x_n} \to 0$, does it follow that $x_n \to \infty$?

2. Let $x_1 > y_1 > 0$ and for every $n \geqslant 1$ let $x_{n+1} = (x_n + y_n)/2$ and $y_{n+1} = 2x_n y_n/(x_n + y_n)$. Show that $x_n > x_{n+1} > y_{n+1} > y_n$. Deduce that (x_n) and (y_n) converge to a common limit. What is that limit?

3. For each $n \in \mathbb{N}$ a closed interval $[x_n, y_n]$ is given. Assume that $[x_m, y_m] \cap [x_n, y_n] \neq \emptyset$ for all $m, n \in \mathbb{N}$. Show that $\bigcap_{n=1}^{\infty}[x_n, y_n] \neq \emptyset$.

4. Give an example of a divergent sequence (x_n) with $x_n - x_{n+1} \to 0$ as $n \to \infty$. Can such a sequence be bounded?

5. Let (x_n) and (y_n) be sequences such that (x_n) is a subsequence of (y_n) and (y_n) is a subsequence of (x_n). Does it follow that $x_n = y_n$ for all n? Does your answer change if we further assume that (x_n) is convergent?

6. Let x be a real or complex number. Assume that every subsequence of a sequence (x_n) has a further subsequence that converges to x. Deduce that (x_n) converges to x.

7. Let (x_n) be a real sequence. Let L be the set of those $x \in \mathbb{R}$ for which there is a subsequence of (x_n) that converges to x. Which of the following subsets of $\mathbb{R}$ can occur as L?

$$\emptyset \qquad \{0\} \qquad \{0,1\} \qquad \mathbb{Z} \qquad \mathbb{Q} \qquad \mathbb{R}$$

Give examples or proofs as appropriate. Show further that if (x_n) is bounded but not convergent then L contains at least two elements.

8. Let $f\colon \mathbb{R} \to (0, \infty)$ be a decreasing function. (That is, if $x \leqslant y$ then $f(x) \geqslant f(y)$.) Define a sequence (x_n) inductively as follows. Set $x_1 = 1$ and $x_{n+1} = x_n + f(x_n)$ for every $n \geqslant 1$. Prove that $x_n \to \infty$.

9. Investigate the convergence of the following series. For each expression that contains the variable z, find all complex numbers z for which the series converges.

$$\sum_n \frac{\sin n}{n^2} \qquad \sum_n \frac{n^2 z^n}{5^n} \qquad \sum_n \frac{(-1)^n}{4+\sqrt{n}} \qquad \sum_n \frac{z^n(1-z)}{n} \qquad \sum_{n\geqslant 3} \frac{n^2}{(\log\log n)^{\log n}}$$

10. The two series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$ and $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \dots$ have the same terms in different orders. Let s_n and, respectively, t_n be the n^{th} partial sums of these series. Set $h_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$. Show that $s_{2n} = h_{2n} - h_n$ and $t_{3n} = h_{4n} - \frac{1}{2}h_{2n} - \frac{1}{2}h_n$. Show that (s_n) converges to a limit s and (t_n) tends to $3s/2$.

11. Show that $\sum_{n\geqslant 2} \frac{1}{n(\log n)^\alpha}$ converges for $\alpha > 1$ and diverges otherwise. Does $\sum_{n\geqslant 3} \frac{1}{n \log n \log\log n}$ converge?

12. Let $x_n > 0$ and $y_n > 0$ for all $n \in \mathbb{N}$. Assume that for some $N \in \mathbb{N}$ we have

$$\frac{x_{n+1}}{x_n} \leqslant \frac{y_{n+1}}{y_n} \qquad \text{for all } n \geqslant N\ .$$

Show that if $\sum y_n$ converges, then so does $\sum x_n$.

13. Can you enumerate $\mathbb{Q}$ as $q_1, q_2, \dots$ so that the series $\sum (q_n - q_{n+1})^2$ is convergent? How about $\sum |q_n - q_{n+1}|$?

14. Assume that $\sum x_n$ is convergent but not absolutely convergent. Show that for all $x \in \mathbb{R}$ there is a permutation $\rho\colon \mathbb{N} \to \mathbb{N}$ such that $\sum_{n=1}^\infty x_{\rho(n)} = x$.

15. Let (x_n) and (y_n) be real sequences.

(i) Suppose $x_n \to 0$ as $n \to \infty$. Show that there is a sequence (ε_n) of signs (*i.e.*, $\varepsilon_n \in \{-1, +1\}$ for all n) such that $\sum \varepsilon_n x_n$ is convergent.

(ii) Suppose $x_n \to 0$ and $y_n \to 0$. Must there be a sequence (ε_n) of signs such that $\sum \varepsilon_n x_n$ *and* $\sum \varepsilon_n y_n$ are both convergent?

16. Let S be a (possibly infinite) set of odd positive integers. Prove that there exists a real sequence (x_n) such that, for each positive integer k, the series $\sum_{n=1}^\infty x_n^k$ converges when k belongs to S and diverges otherwise.