ANALYSIS 1 EXAMPLES SHEET 2

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1. Let (a_n) and (b_n) be two real sequences. Suppose that (a_n) is a subsequence of (b_n) and (b_n) is a subsequence of (a_n). Suppose also that (a_n) converges. Does it follow that they are the same sequence?

2. Let $H : \mathbb{R} \to \mathbb{R}$ be defined as follows: if $x < 0$ then $H(x) = 0$ and if $x \geq 0$ then $H(x) = 1$. Prove carefully that H is not continuous (i) by directly using the definition of continuity and (ii) by using the sequence definition.

3. Suppose that $f(x) \to \ell$ as $x \to a$ and $g(y) \to k$ as $y \to \ell$. Does it follow that $g(f(x)) \to k$ as $x \to a$?

4. For each natural number n, let $f_n : [0,1] \to [0,1]$ be a continuous function, and for each n let h_n be defined by $h_n(x) = \max\{f_1(x), \dots, f_n(x)\}$. Show that for each n the function h_n is continuous on $[0,1]$. Must the function h defined by $h(x) = \sup\{f_n(x) : n \in \mathbb{N}\}$ be continuous?

5. Let $g : [0,1] \to [0,1]$ be a continuous function. Prove that there exists some $c \in [0,1]$ such that $g(c) = c$. Such a c is called a *fixed point* of g.

Give an example of a bijection $h : [0,1] \to [0,1]$ with no fixed point.

Give an example of a continuous bijection $p : (0,1) \to (0,1)$ with no fixed point.

6. Prove that the function $q(x) = 2x^5 + 3x^4 + 2x + 16$ (defined on the reals) takes the value 0 exactly once, and that the number where it takes that value is somewhere in the interval $[-2, -1]$.

7. Prove rigorously that there are exactly nine solutions to the simultaneous equations $x = 1000(y^3 - y)$ and $y = 1000(x^3 - x)$. That is, prove that there are exactly nine ordered pairs (x, y) such that the two equations are satisfied.

8. Let $f : [0,1] \to \mathbb{R}$ be continuous, with $f(0) = f(1) = 0$. Suppose that for every $x \in (0,1)$ there exists $\delta > 0$ such that both $x + \delta$ and $x - \delta$ belong to $(0,1)$ and $f(x) = \frac{1}{2}(f(x - \delta) + f(x + \delta))$. Prove that $f(x) = 0$ for every $x \in [0,1]$.

9. Define a function $f : \mathbb{R} \to \mathbb{R}$ as follows. If x is irrational, then $f(x) = 0$, while if x is rational, then $f(x) = 1/q$, where q is the denominator of x. (That is, $x = p/q$, with p and q coprime integers and $q > 0$.) Prove that f is continuous at every irrational and discontinuous at every rational.

10. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Which of the following statements are always true and which are sometimes false?

(i) If f is increasing, then $f'(x) \geq 0$ for every $x \in (a, b)$.
(ii) If $f'(x) \geq 0$ for every $x \in (a, b)$, then f is increasing.
(iii) If f is strictly increasing, then $f'(x) > 0$ for every $x \in (a, b)$.
(iv) If $f'(x) > 0$ for every $x \in (a, b)$, then f is strictly increasing.

11. (i) Let $g : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $g(0) = g'(0) = 0$ and $g''(0)$ exists and is positive. Prove that there exists $x > 0$ such that $g(x) > 0$.

(ii) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f(0) = 0$, and $f''(0)$ exists and is positive. Prove that there exists $x > 0$ such that $f(2x) > 2f(x)$.

12. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable everywhere. Prove that if $f'(x) \to \ell$ as $x \to \infty$, then $f(x)/x \to \ell$. If $f(x)/x \to \ell$ as $x \to \infty$, does it follow that $f'(x) \to \ell$?

13. Find a function $f : \mathbb{R} \to \mathbb{R}$ that takes every value in every interval. That is, for every $a < b$ and every t there should exist $x \in (a, b)$ such that $f(x) = t$.

14. Let $f : \mathbb{R} \to \mathbb{R}$ be a function that has the intermediate value property: that is, if $f(a) < c < f(b)$ then there exists $x \in (a, b)$ such that $f(x) = c$. Suppose also that for every rational r the set $S_r = \{x : f(x) = r\}$ is closed. (This means that if (x_n) is any convergent sequence in S_r, then its limit also belongs to S_r.) Prove that f is continuous.