ANALYSIS 1 EXAMPLES SHEET 1

Lent Term 2015 W. T. G.

1. Let (a_n) and (b_n) be two real sequences. Suppose that (a_n) is a subsequence of (b_n) and (b_n) is a subsequence of (a_n). Does it follow that they are the same sequence?

2. For each positive integer k let $a_{2^k} = 1$ and for every n that is not a power of 2, let $a_n = 0$. Prove directly from the definition of convergence that the sequence (a_n) does not converge.

3. Let (a_n) be a real sequence. We say that $a_n \to \infty$ if for every K there exists N such that for every $n \geq N$ we have $a_n \geq K$.
(i) Write down a similar definition for $a_n \to -\infty$.
(ii) Show that $a_n \to -\infty$ if and only if $-a_n \to \infty$.
(iii) Suppose that no a_n is 0. Prove that if $a_n \to \infty$, then $\frac{1}{a_n} \to 0$.
(iv) Again suppose that no a_n is 0. If $\frac{1}{a_n} \to 0$, does it follow that $a_n \to \infty$?

4. Let $a_1 > b_1 > 0$ and for every $n \geq 1$ let $a_{n+1} = (a_n + b_n)/2$ and let $b_{n+1} = 2a_n b_n/(a_n + b_n)$. Show that $a_n > a_{n+1} > b_{n+1} > b_n$. Deduce that the two sequences converge to a common limit. What is that limit?

5. Let $(a_1, b_1) \supset (a_2, b_2) \supset \dots$ be a nested sequence of non-empty open intervals. Must $\bigcap_{n=1}^{\infty} (a_n, b_n)$ be non-empty? If not, then find a (non-trivial) additional condition that guarantees that the intersection is non-empty.

6. (i) Let (a_n) be a real sequence that is bounded but that does not converge. Prove that it has two convergent subsequences with different limits.
(ii) Prove that every real sequence has a subsequence that converges or tends to $\pm\infty$.

7. Let a be a real number and let (a_n) be a sequence such that every subsequence of (a_n) has a further subsequence that converges to a. Prove that $a_n \to a$.

8. Let (a_n) be a Cauchy sequence. Prove that (a_n) has a subsequence (a_{n_k}) such that $|a_{n_p} - a_{n_q}| < 2^{-p}$ whenever $p \leq q$.

9. Let $f : \mathbb{R} \to (0, \infty)$ be a decreasing function. (That is, if $x < y$ then $f(x) \geq f(y)$.) Define a sequence (a_n) inductively by $a_1 = 1$ and $a_{n+1} = a_n + f(a_n)$ for every $n \geq 1$. Prove that $a_n \to \infty$.

10. Investigate the convergence of the following series. For each expression that contains the variable z, find all complex numbers z for which the series converges.

$$\sum_n \frac{\sin n}{n^2} \qquad \sum_n \frac{n^2 z^n}{5^n} \qquad \sum_n \frac{(-1)^n}{4+\sqrt{n}} \qquad \sum_n \frac{z^n(1-z)}{n} \qquad \sum_{n\geq 3} \frac{n^2}{(\log\log n)^{\log n}}$$

11. The two series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$ and $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \dots$ have the same terms but in different orders. Let S_n and T_n be the partial sums to n terms. Let $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$. Show that $S_{2n} = H_{2n} - H_n$ and $T_{3n} = H_{4n} - \frac{1}{2}H_{2n} - \frac{1}{2}H_n$. Show that the sequence (S_n) converges to a limit S and that $T_n \to 3S/2$.

12. Prove that $\sum_n \frac{1}{n(\log n)^\alpha}$ converges if $\alpha > 1$ and diverges otherwise. Does the series $\sum_n \frac{1}{n \log n \log\log n}$ converge?

13. Let (a_n) be a sequence of positive real numbers such that $\sum_n a_n$ diverges. Prove that there exists a sequence (b_n) of positive real numbers such that $b_n/a_n \to 0$, but $\sum_n b_n$ is still divergent.

14. Let x be a real number and let $\sum_n a_n$ be a series that converges but that does not converge absolutely. Prove that the terms can be reordered so that the series converges to x. That is, show that there is a bijection $\pi : \mathbb{N} \to \mathbb{N}$ such that $\sum_n a_{\pi(n)} = x$.

15. For every positive integer k write $\log_k(x)$ for $\log\log\dots\log(x)$, where the logarithm has been taken k times. (Thus, $\log_1(x) = \log x$, $\log_2(x) = \log\log x$, and so on.) Define a function $f : \mathbb{N} \to \mathbb{R}$ by taking $f(n)$ to be $n \log n \log_2 n \dots \log_{k(n)} n$, where $k(n)$ is the largest integer such that $\log_{k(n)} n \geq 1$. Does the series $\sum_n \frac{1}{f(n)}$ converge?

16. Can the open interval $(0, 1)$ be written as a union of disjoint closed intervals of positive length?

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